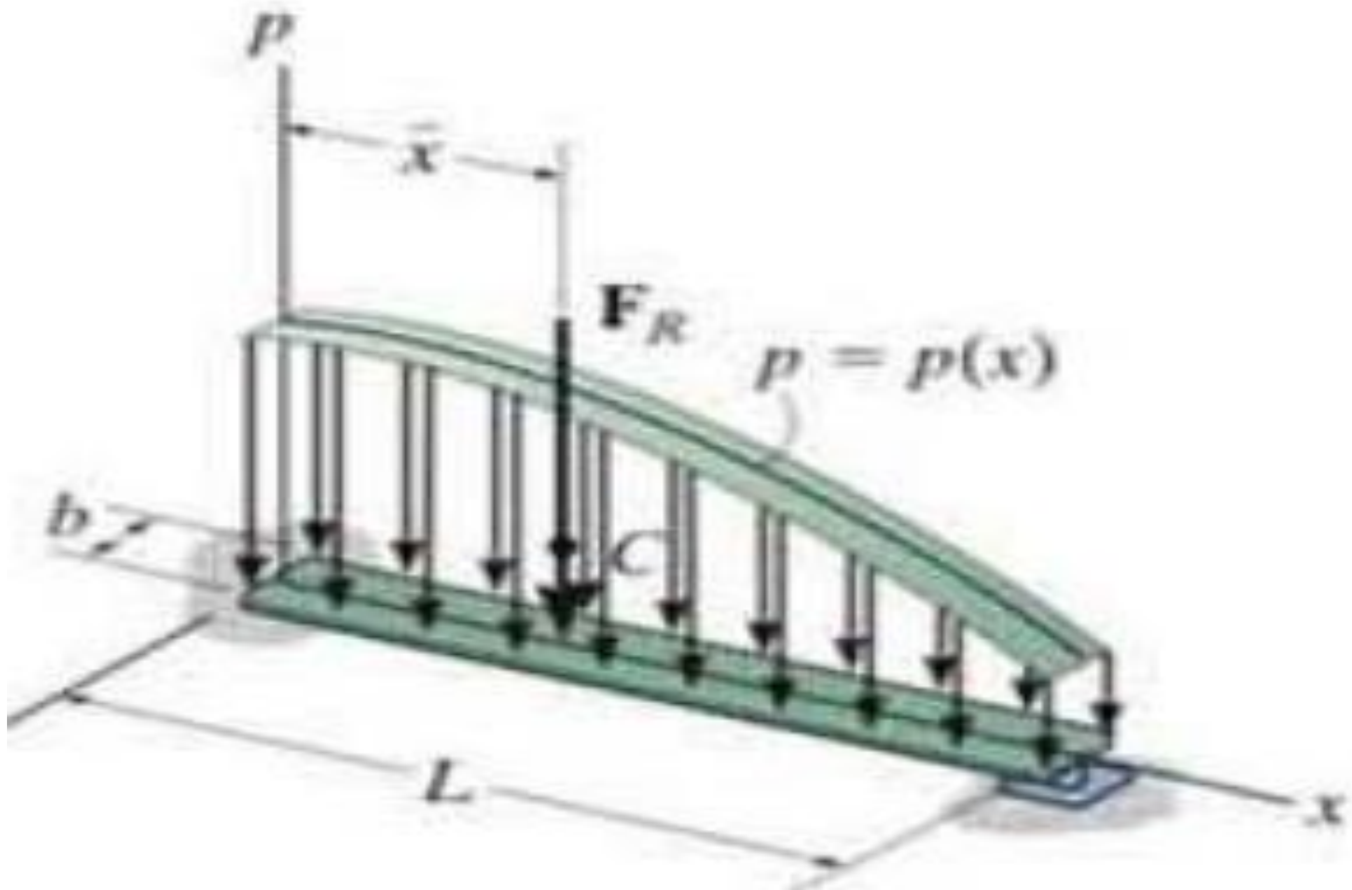
CHAPTER 5

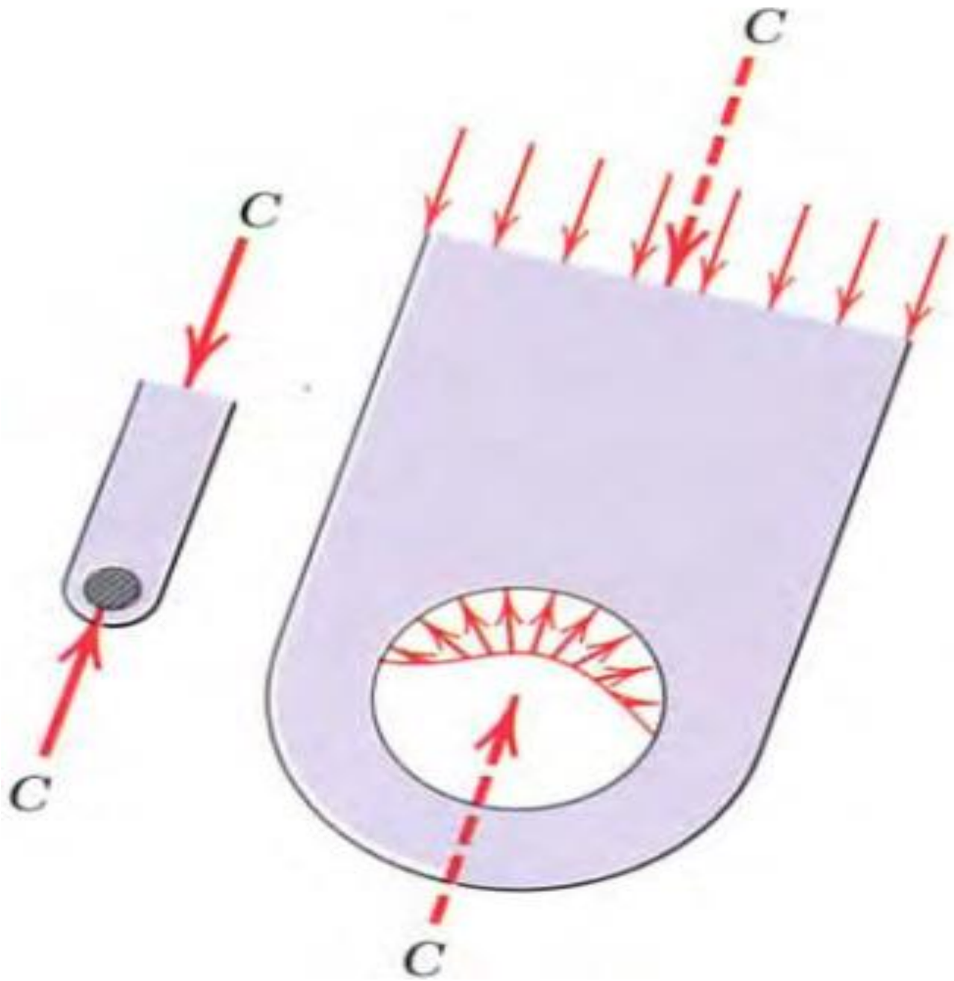
Distributed Forces

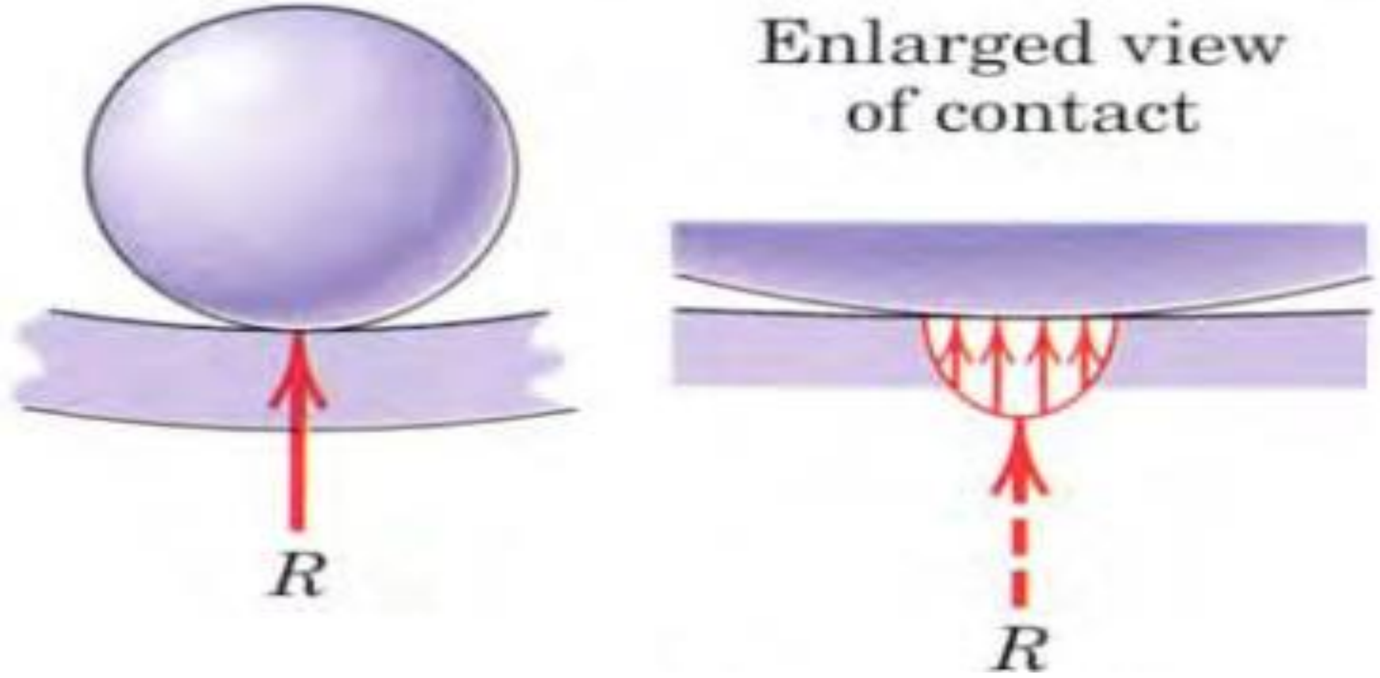
Introduction

- Sometimes, a body may be *subjected* to a *loading* that is *distributed* over its *surface*. For example, the *pressure* of the *wind* on the face of a sign, the *pressure* of *water* within a tank, or the *weight* of *sand* on the floor of a storage container, are all *distributed loadings*.

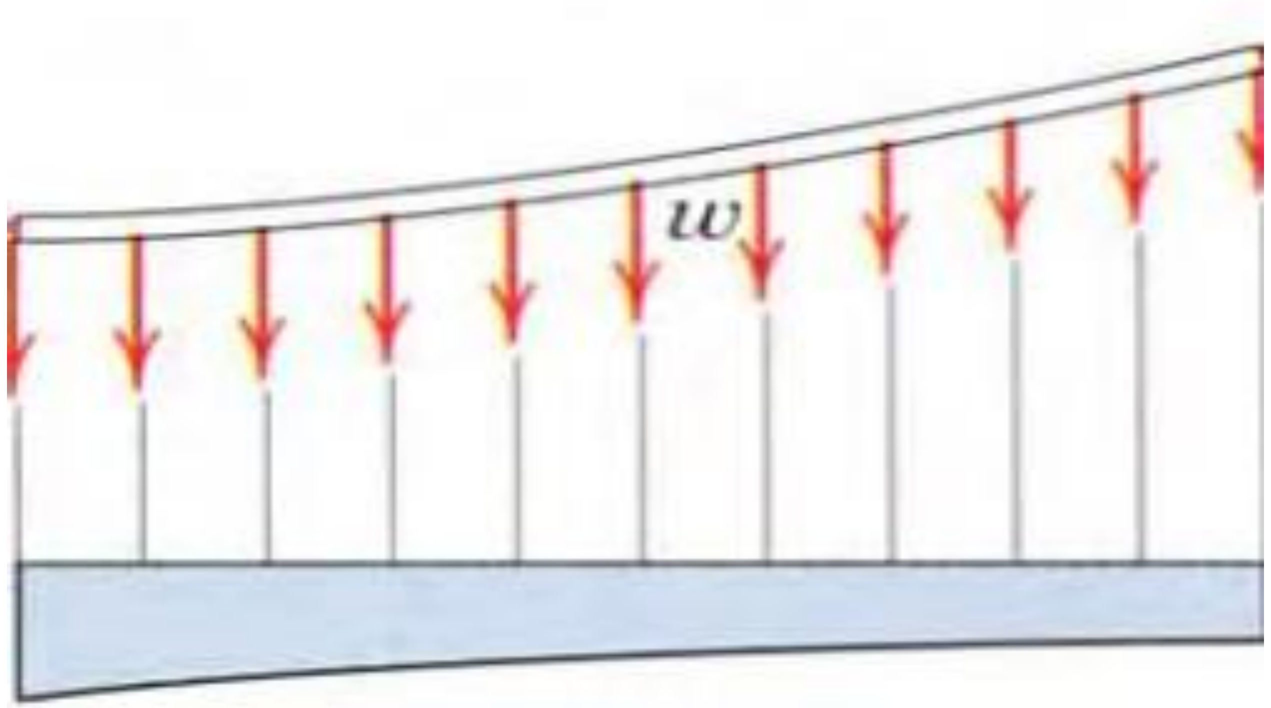

b
R


p
x̄
F_R
p = p(x)
b
C
L
x

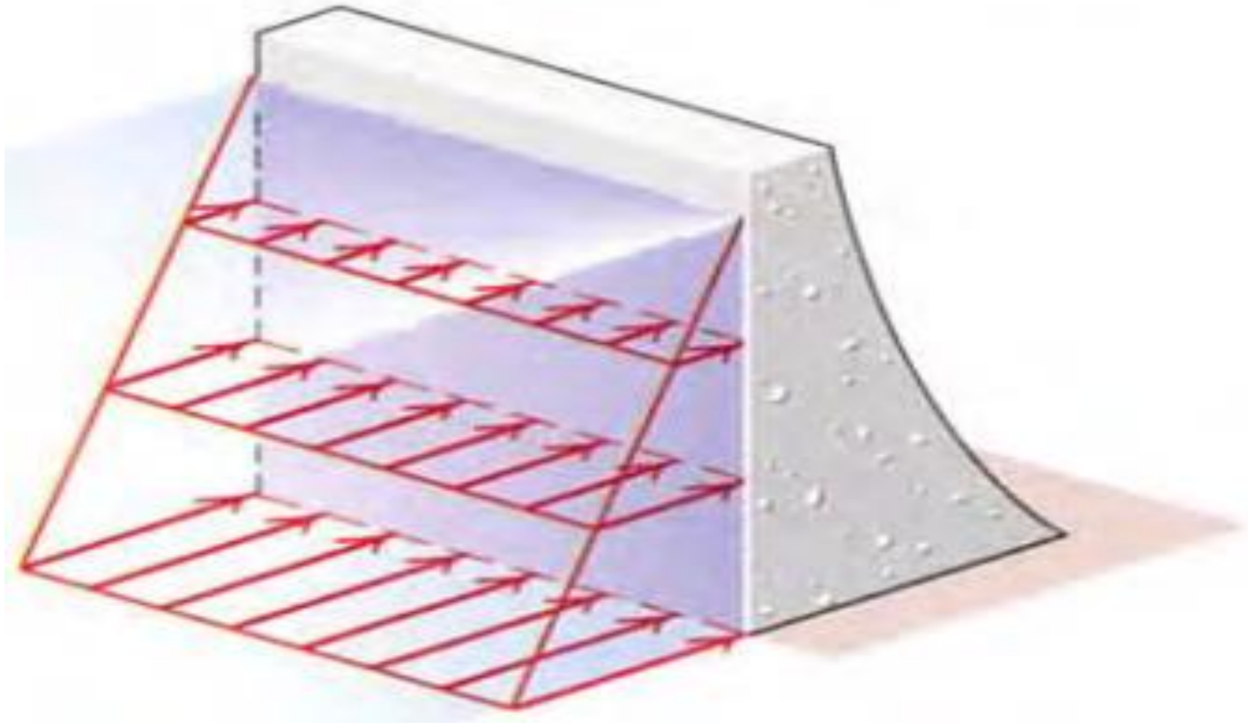

C
C
C
C


Enlarged view
of contact
R
R

- *Line Distribution*: when a *force* is distributed *along a line*, as in the continuous vertical load supported by a suspended cable. The loading is expressed as *force per unit length* of line (N/m).



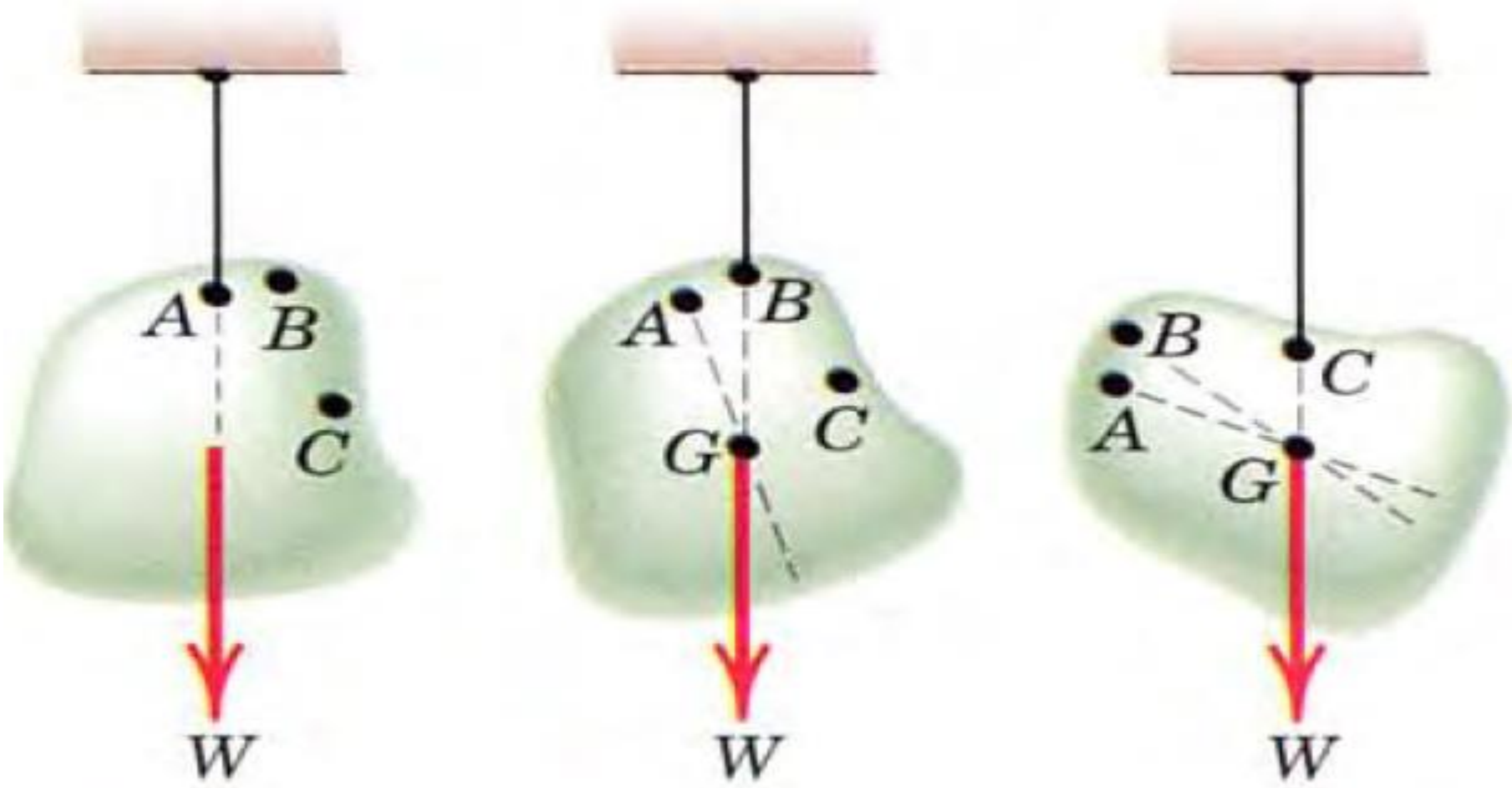
- *Area Distribution*: when a *force* is distributed *over an area*, as with the hydraulic pressure of water against the inner face of a section of dam. The intensity is expressed as *force per unit area* (N/m^2).



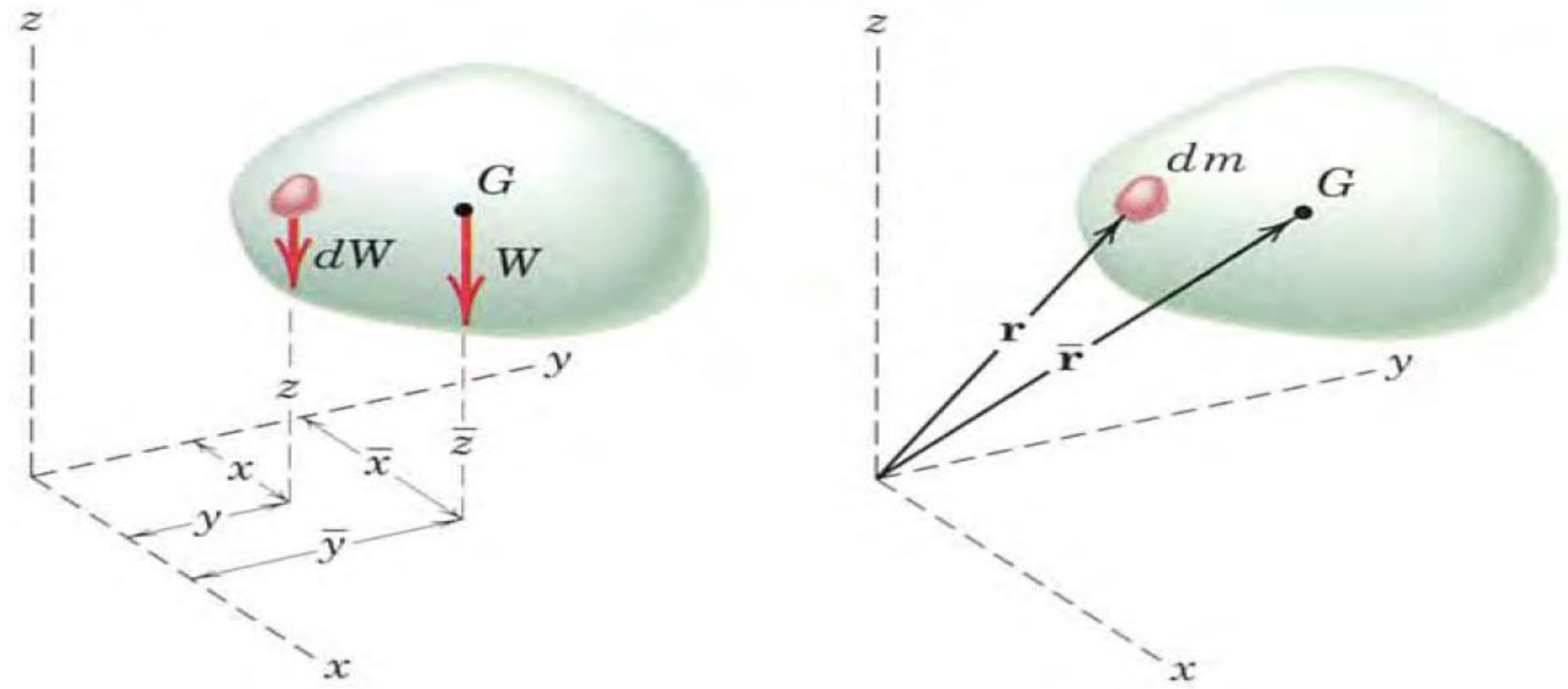
- *Volume Distribution*: a *force* which is distributed *over the volume* of a body is called a body force. The most common *body force* is the force of *gravitational attraction*, which acts on all elements of mass in a body. The intensity of *gravitational force* is the *specific weight* "ρg" [$(kg/m^3)(m/s^2)=N/m^3$].

Center of Mass

- Consider a *3D body* of any size and shape, having a mass *m*. If we *suspend* the body, as shown in figure, from *any point A*, the body will be in equilibrium. We *repeat* the experiment by *suspending* the body from other points *B* and *C*, and each instance we *mark* the *line of action* of the resultant force. These *lines of action* will be *concurrent* at a single point *center of gravity* "*G*" of the body.


A
B
C
W
A
B
C
G
W
B
C
A
G
W

- To determine *mathematically* the location of the *center of gravity* of any body, we *apply* the *principle of moments* to the parallel system of gravitational forces to locate its resultant.

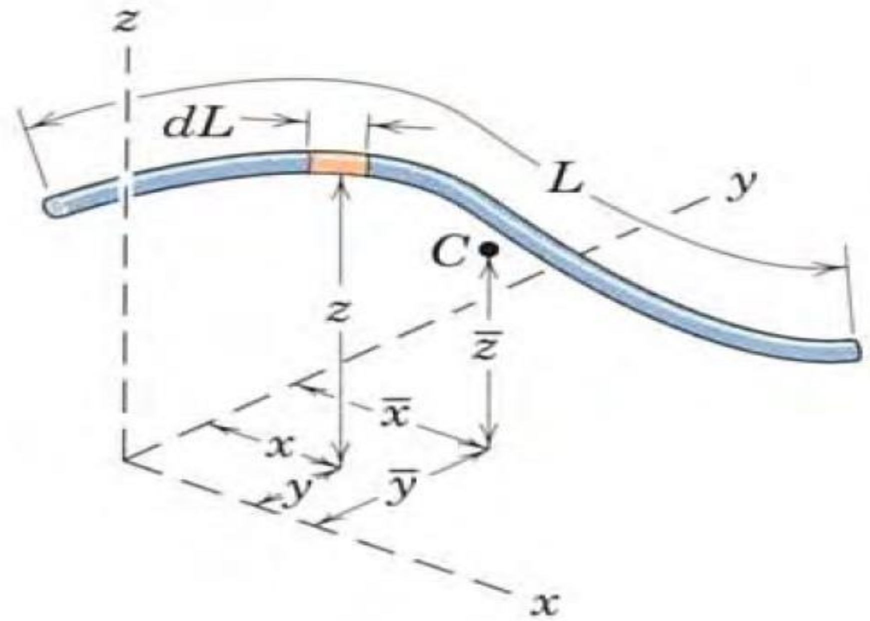


- The *moment* of the resultant *gravitational force* W about any axis equals the sum of the *moments* about the same axis of the *gravitational forces* dW acting on *all particles* treated as *infinitesimal* elements of the body.

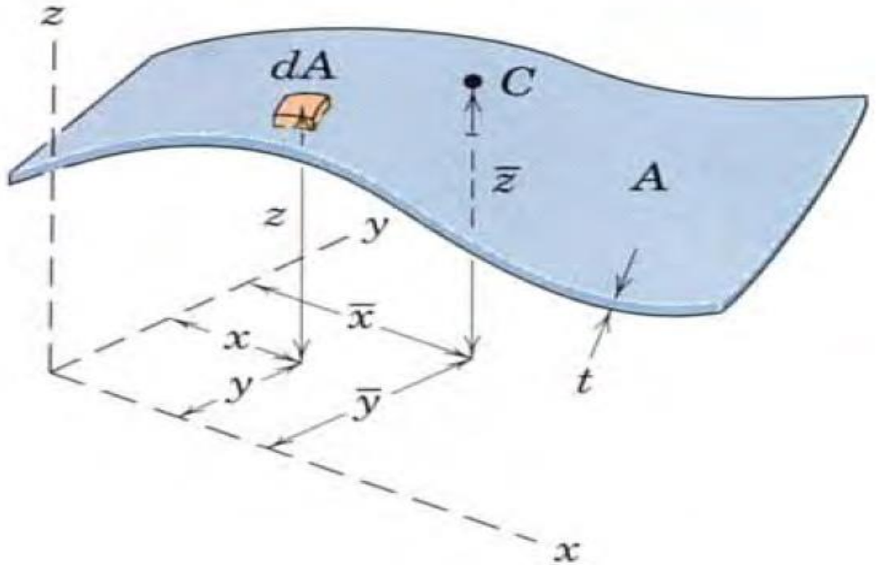
$$\bar{x} = \frac{\int x dW}{W} \quad , \quad \bar{y} = \frac{\int y dW}{W} \quad , \quad \bar{z} = \frac{\int z dW}{W}$$

Centroids of Lines, Areas, and Volumes

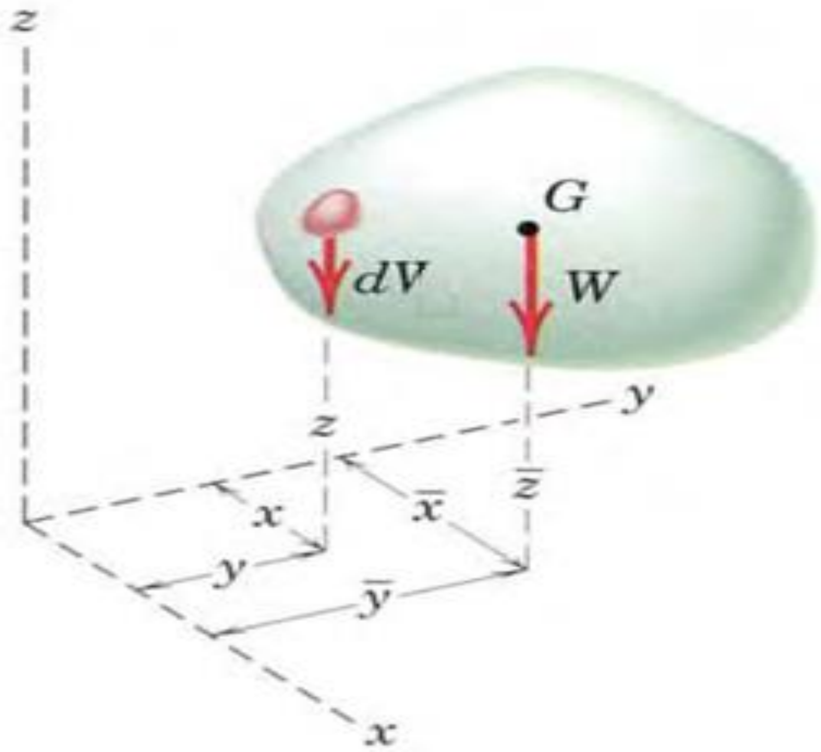
- When the *density ρ* of a body is *uniform*, it will be a constant factor. So, the *expressions* that remain define a purely *geometrical property* of the body, since any reference to its *mass* properties has *disappeared*. The term *centroid* is used when the *calculation* concerns a *geometrical shape* only.



$$\bar{x} = \frac{\int x dL}{L} \quad , \quad \bar{y} = \frac{\int y dL}{L} \quad , \quad \bar{z} = \frac{\int z dL}{L}$$



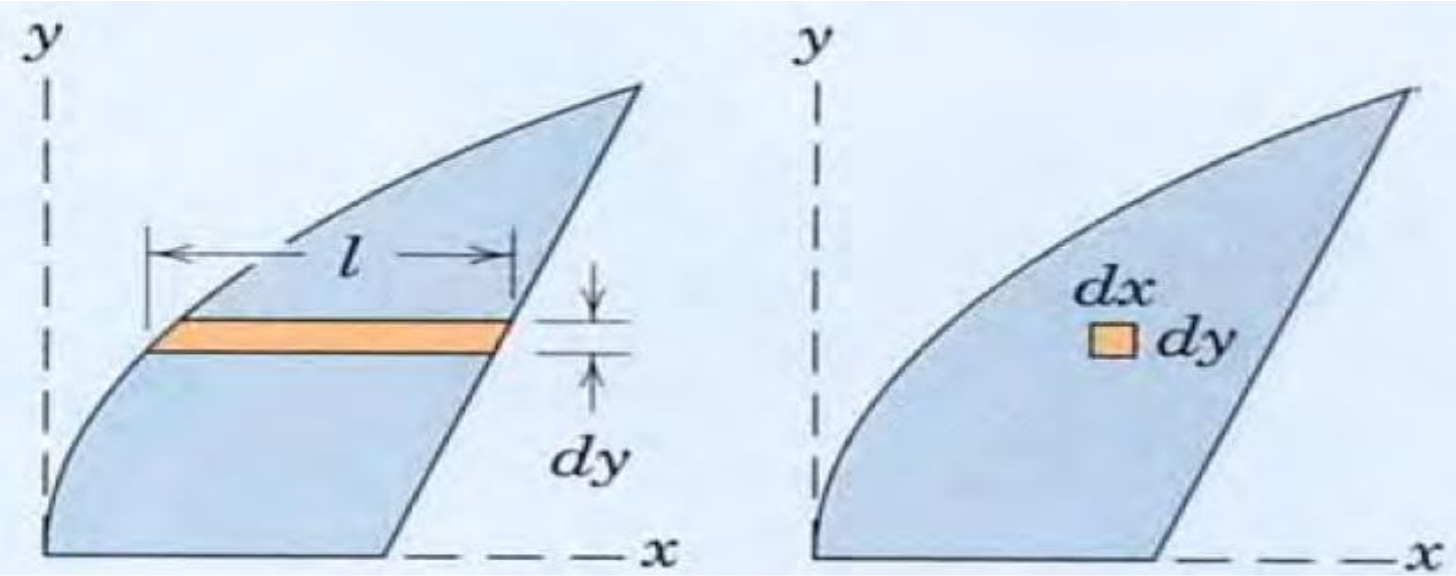
$$\bar{x} = \frac{\int x dA}{A} \quad , \quad \bar{y} = \frac{\int y dA}{A} \quad , \quad \bar{z} = \frac{\int z dA}{A}$$

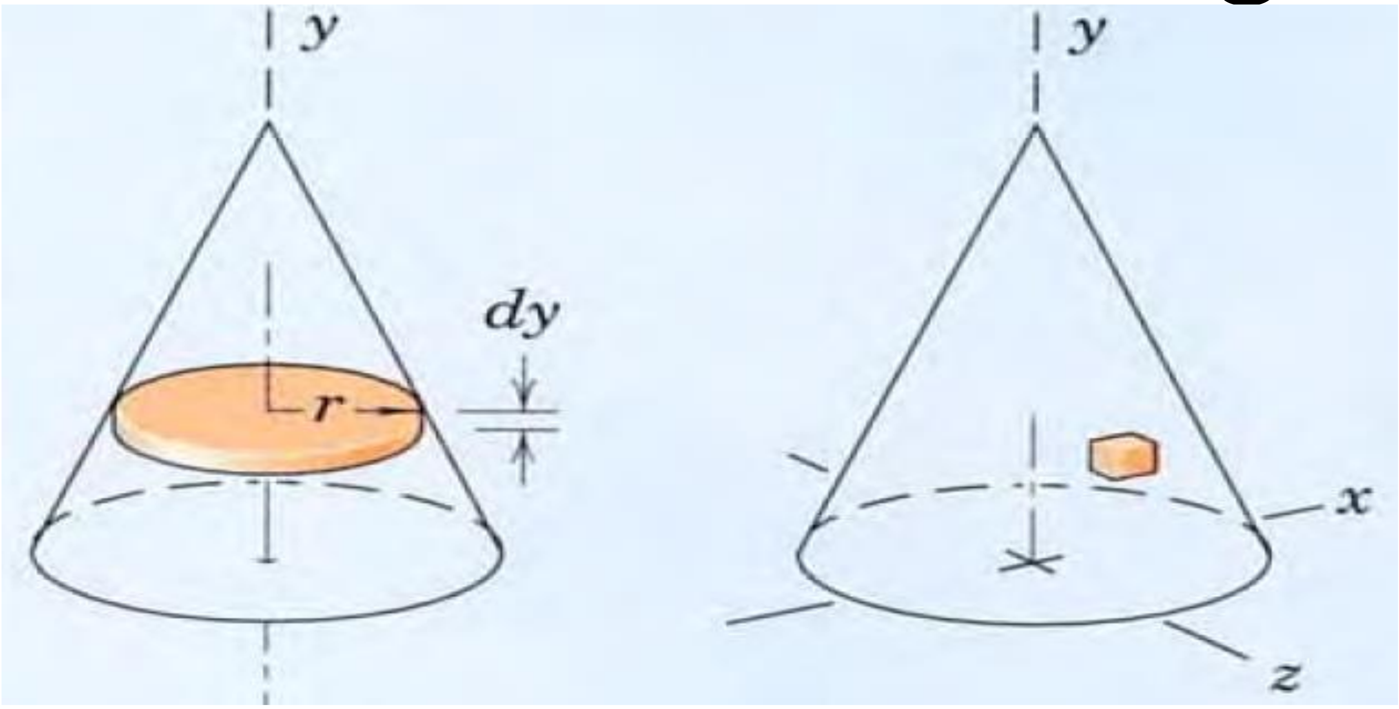


$$\bar{x} = \frac{\int x dV}{V} \quad , \quad \bar{y} = \frac{\int y dV}{V} \quad , \quad \bar{z} = \frac{\int z dV}{V}$$

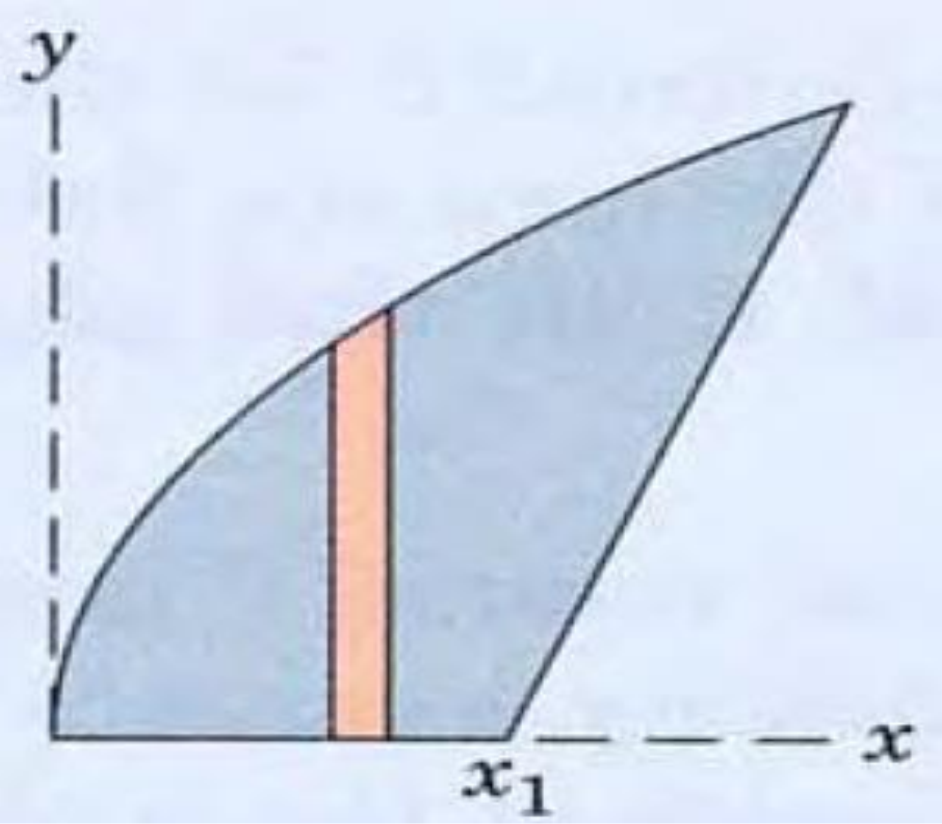
Choice of element for integration

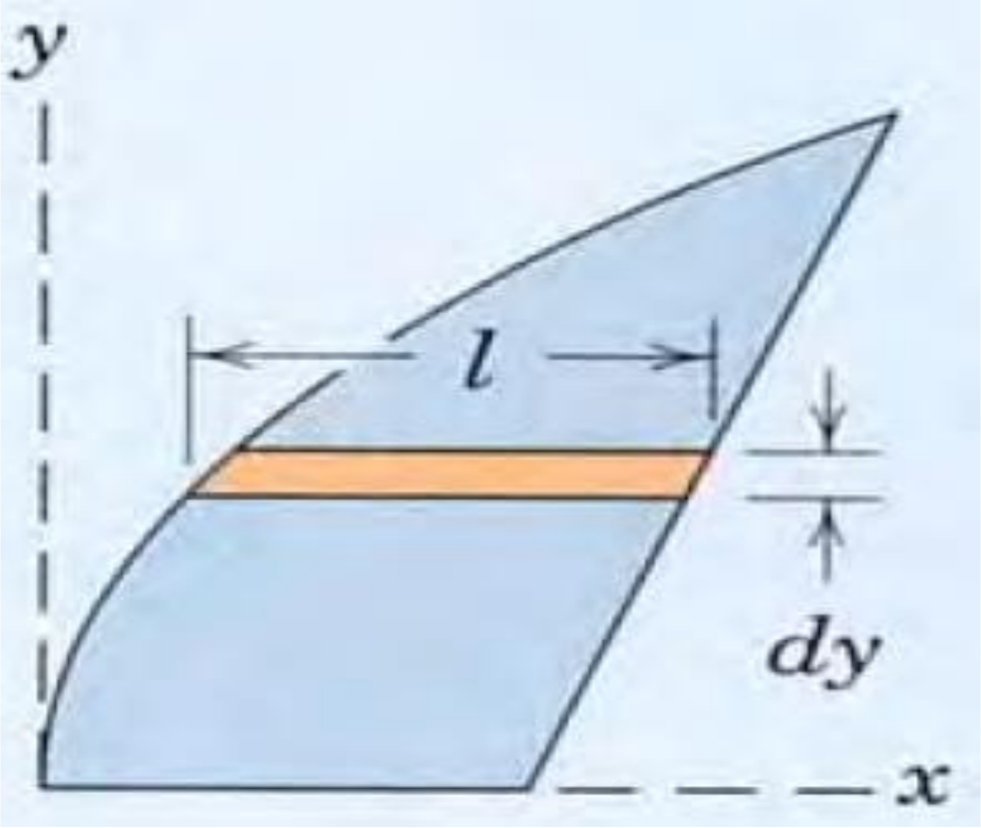
1. *Order of element*: Whenever possible, a *first-order* differential *element* should be *selected* in preference to a *higher-order* element so that only one integration will be required to cover the entire figure.



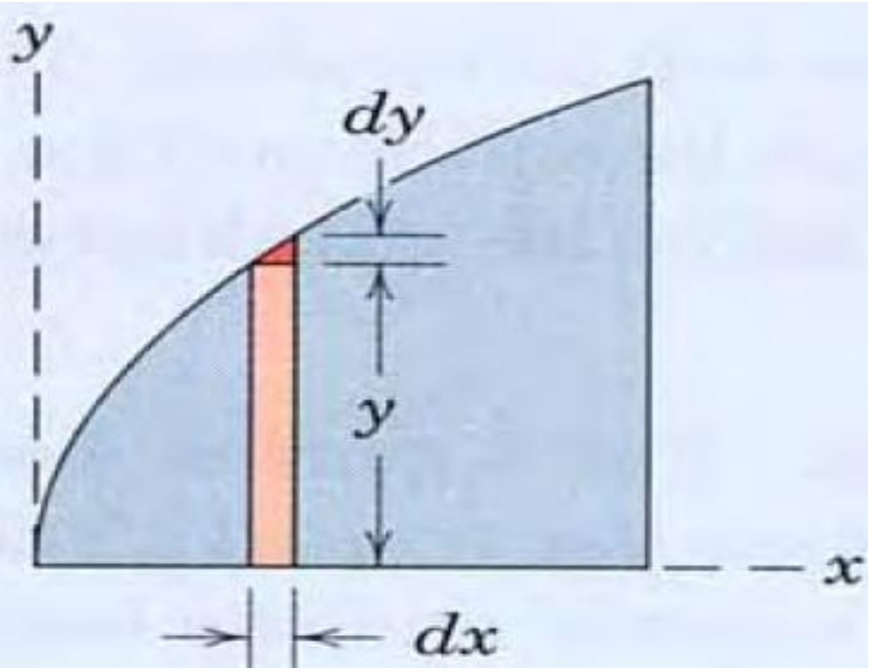


2. *Continuity*: Whenever possible, we choose *an element* which can be integrated in *one continuous operation* to cover the figure.

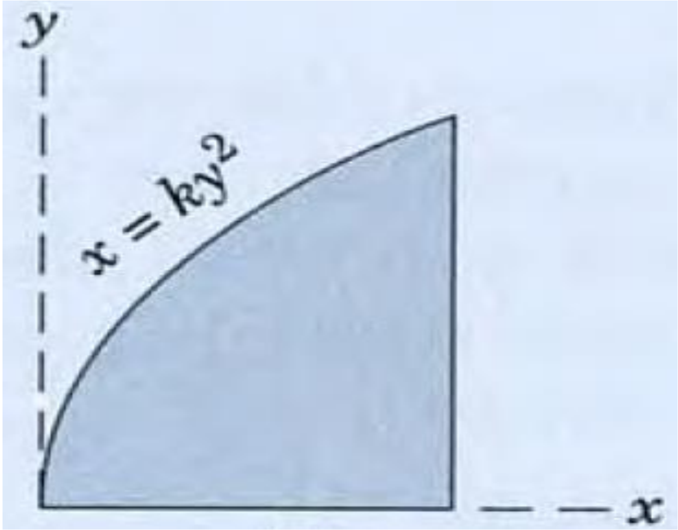


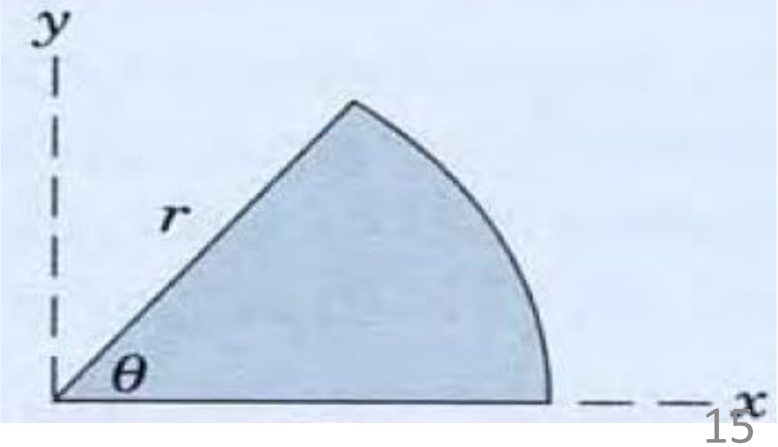


3. *Discarding higher-order terms*: Higher-order terms may always be dropped compared with lower-order terms.

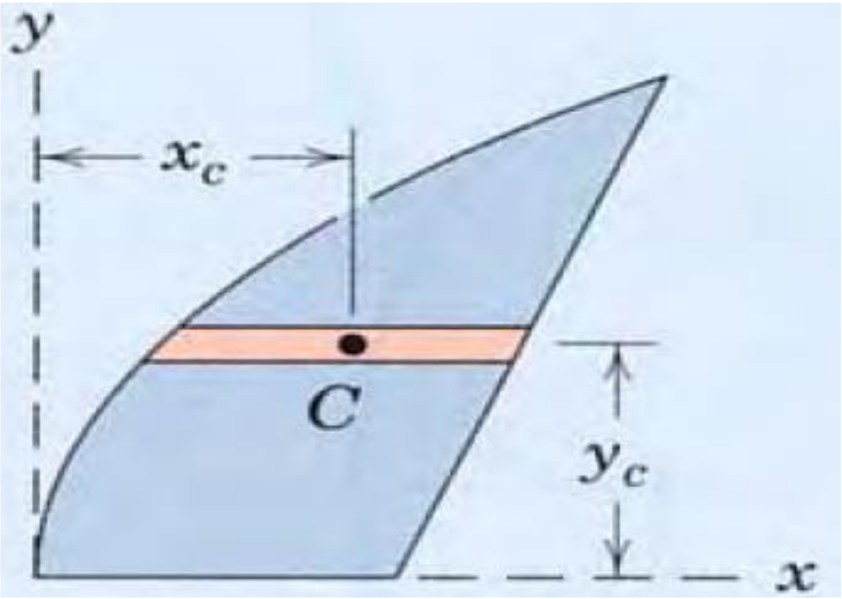


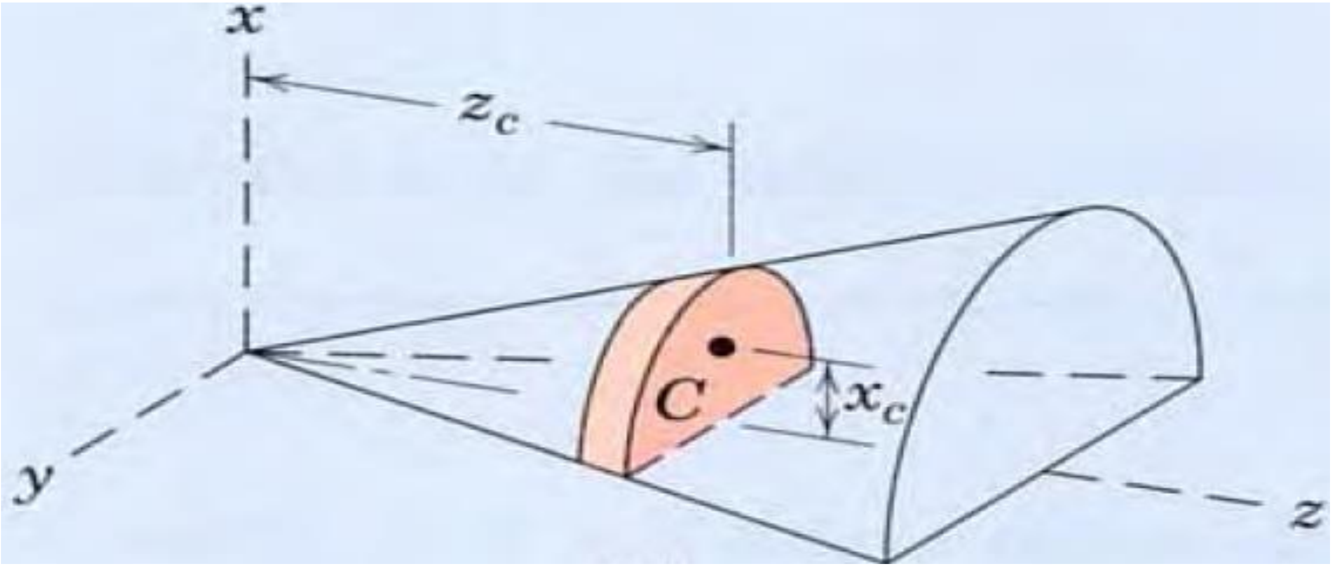
4. *Choice of coordinate*: As a general rule, we choose the coordinate system which best matches the boundaries of the figure.





5. *Centrodial coordinate of element*: When a first or second order *differential element* is adopted, it is essential to use the coordinate of the *centroid* of the *element* for the *moment arm* in setting up the moment of the differential element.



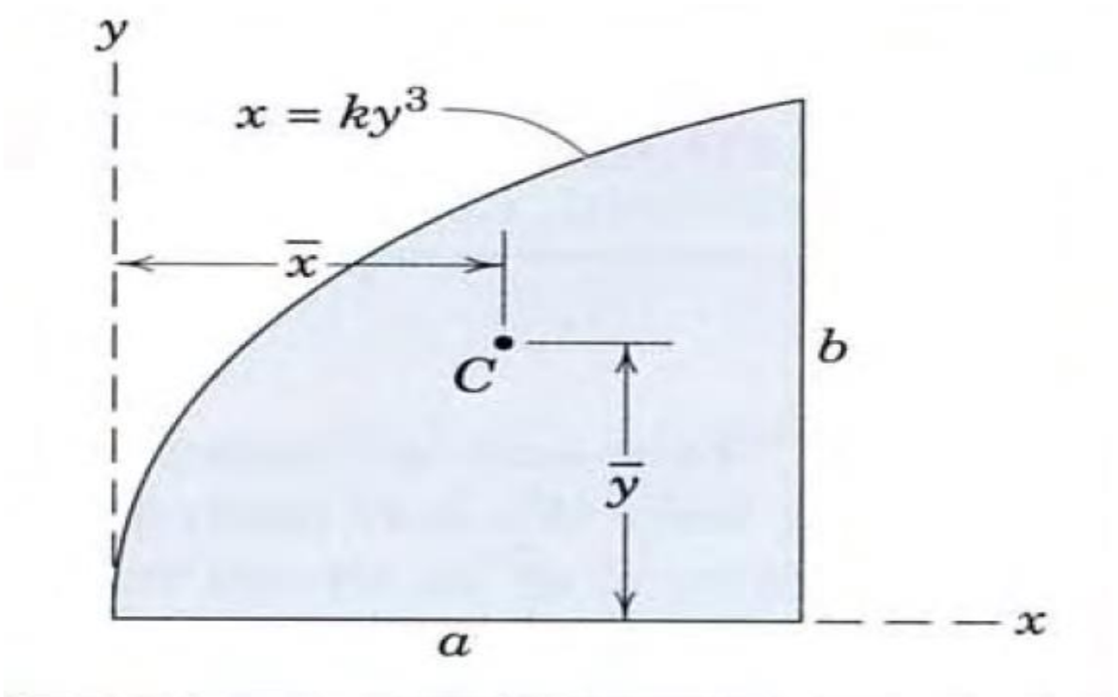


$$\bar{x} = \frac{\int x_c dA}{A} \quad , \quad \bar{y} = \frac{\int y_c dA}{A} \quad , \quad \bar{z} = \frac{\int z_c dA}{A}$$

$$\bar{x} = \frac{\int x_c dV}{V} \quad , \quad \bar{y} = \frac{\int y_c dV}{V} \quad , \quad \bar{z} = \frac{\int z_c dV}{V}$$

Example 1

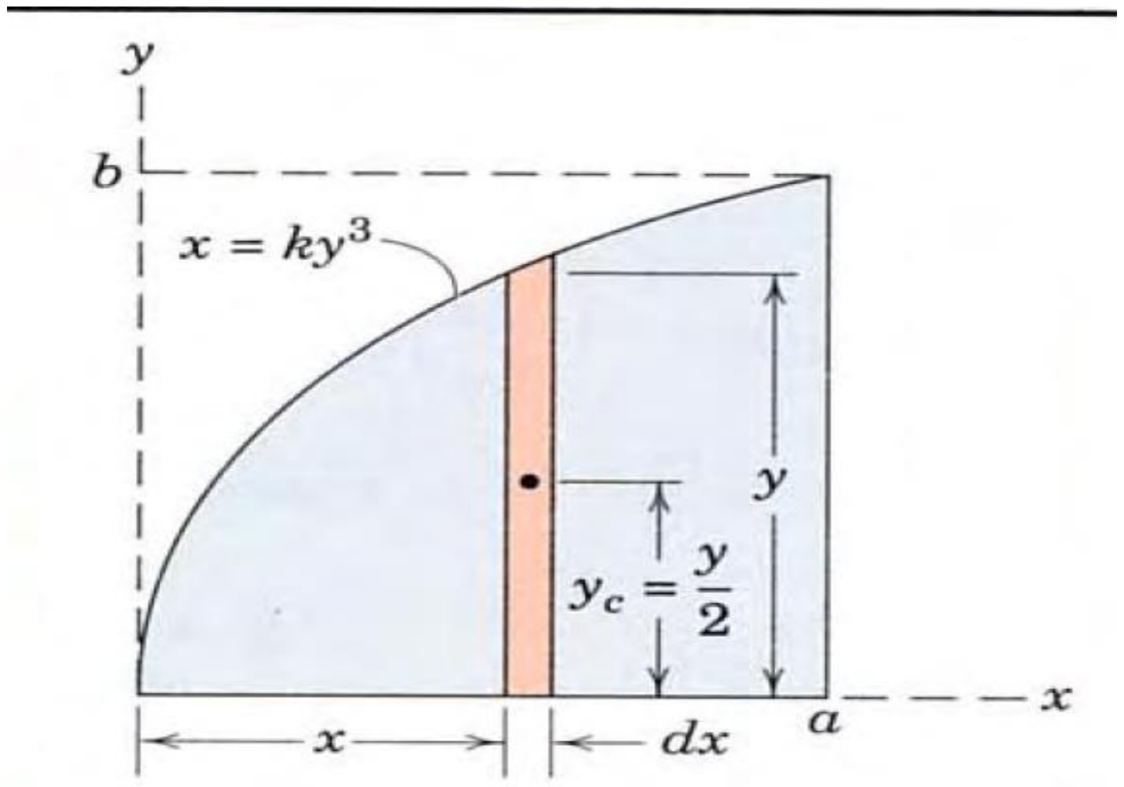
- Find the cenroid of the area under the curve $x=ky^3$.



Sol 1

$$dA = y\,dx \qquad y = (x/k)^{1/3} \qquad k = a/b^3$$

$$A = \int_0^a y\,dx \qquad A = \frac{3ab}{4}$$



$$[A\bar{x} = \int x_c\,dA] \qquad \bar{x}\int_0^a y\,dx = \int_0^a xy\,dx$$

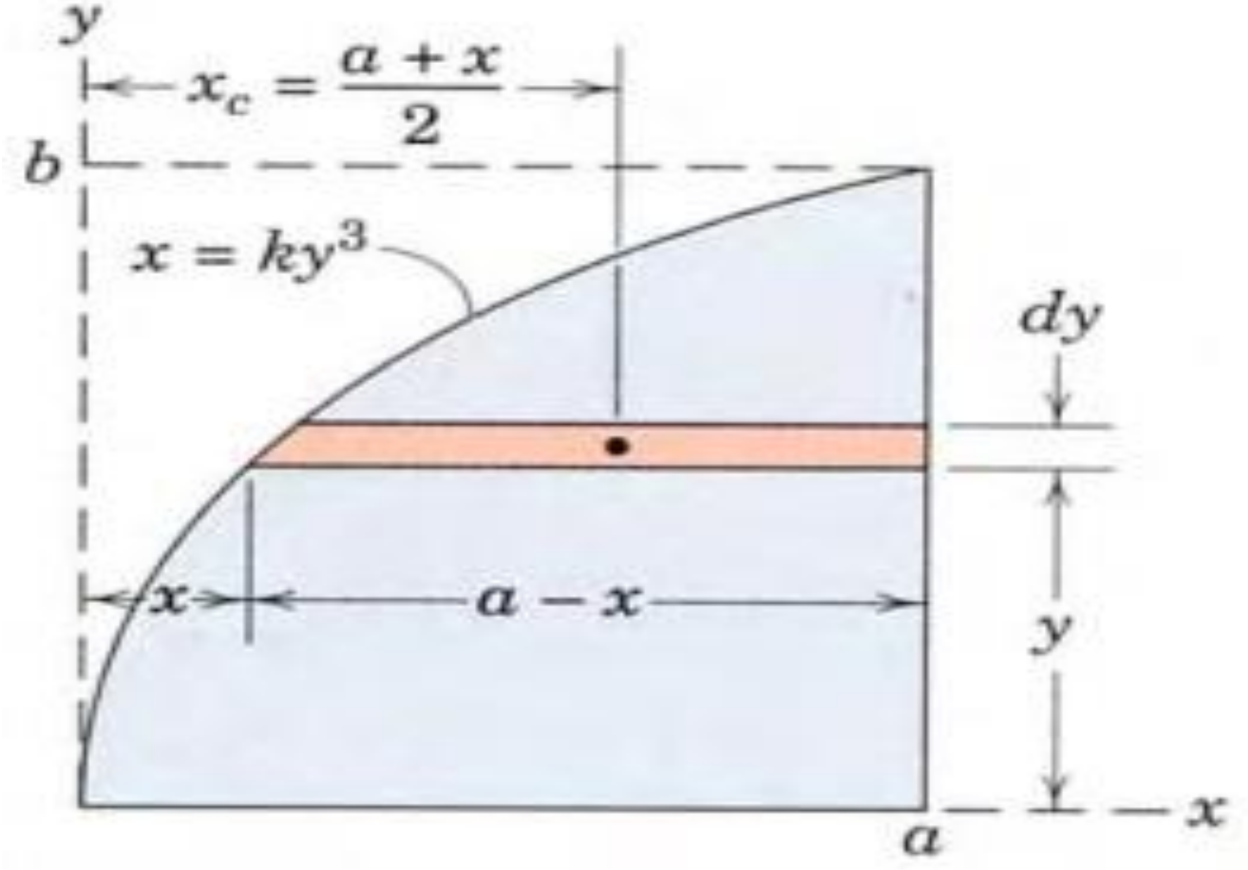
$$A\bar{x} = \frac{3ab}{4}\bar{x} = \frac{3a^2b}{7} \qquad \bar{x} = \tfrac{4}{7}a$$

$$[A\bar{y} = \int y_c\,dA] \qquad \frac{3ab}{4}\bar{y} = \int_0^a \left(\frac{y}{2}\right) y\,dx$$

$$y = b(x/a)^{1/3}$$

$$\frac{3ab}{4}\bar{y} = \frac{3ab^2}{10} \qquad \bar{y} = \tfrac{2}{5}b$$

Sol 1   2. way



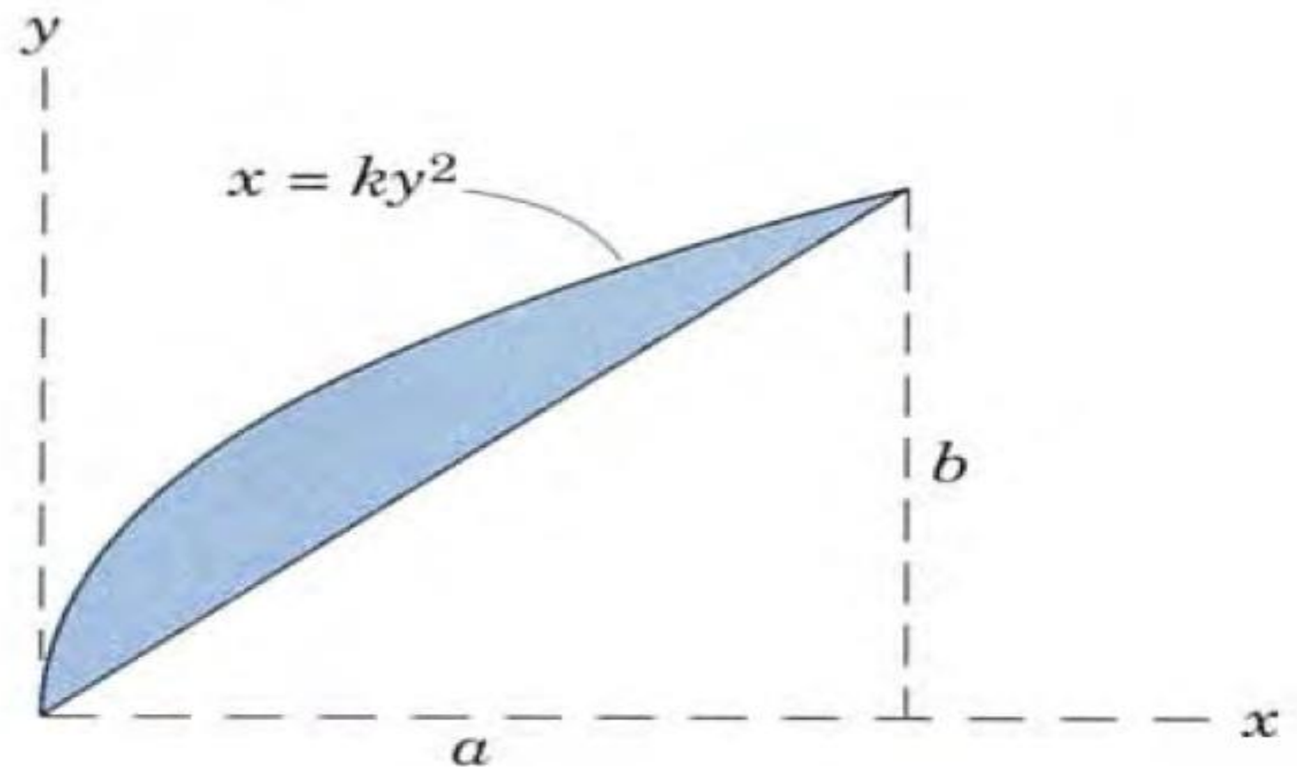
$$x_c = x + \tfrac{1}{2}(a - x) = (a + x)/2,$$

$$\left[A\bar{x} = \int x_c \, dA\right] \qquad \bar{x}\int_0^b (a - x)\, dy = \int_0^b \left(\frac{a + x}{2}\right)(a - x)\, dy$$

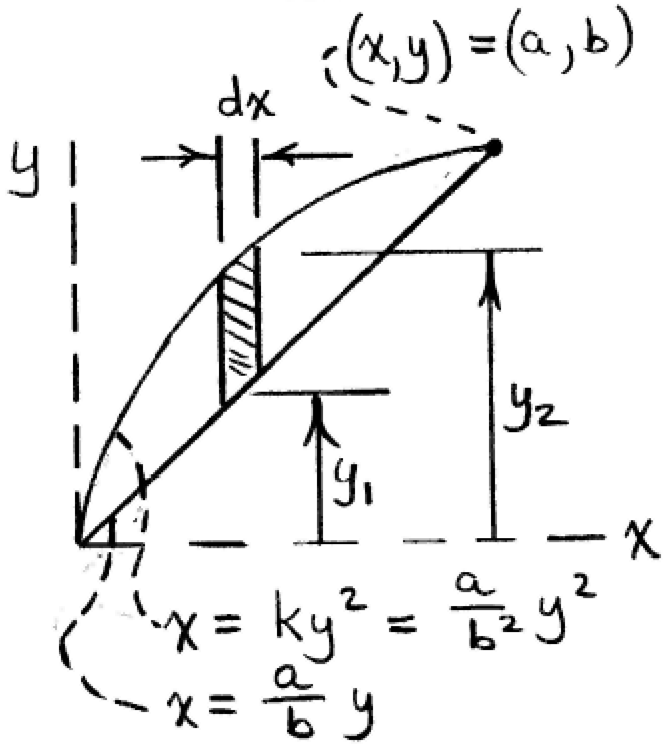
$$\left[A\bar{y} = \int y_c \, dA\right] \qquad \bar{y}\int_0^b (a - x)\, dy = \int_0^b y(a - x)\, dy$$

Example 2

- Find the cenroid of the shaded area.



Sol 2



$$A = \int_0^a (y_2 - y_1)\,dx$$
$$= \int_0^a \left(b\sqrt{\frac{x}{a}} - x\frac{b}{a}\right)dx$$
$$= b\left[\frac{1}{\sqrt{a}}\frac{2x^{3/2}}{3} - \frac{1}{2a}x^2\right]_0^a$$
$$= \frac{ab}{6}$$

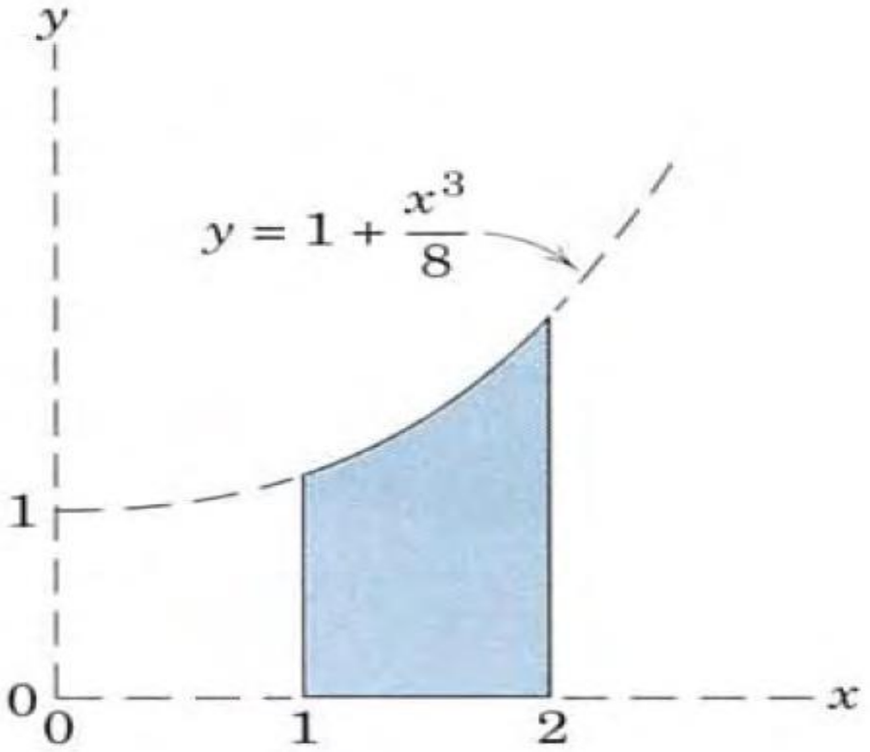
$$\int x_c\,dA = \int_0^a x(y_2 - y_1)\,dx = \int_0^a \left[\frac{b}{\sqrt{a}}x^{3/2} - \frac{b}{a}x^2\right]dx$$
$$= b\left[\frac{2x^{5/2}}{5\sqrt{a}} - \frac{x^3}{3a}\right]_0^a = \frac{a^2b}{15}$$

$$\bar{x} = \frac{\int x_c\,dA}{A} = \frac{ab^2/15}{ab/6} = \underline{\frac{2}{5}a}$$

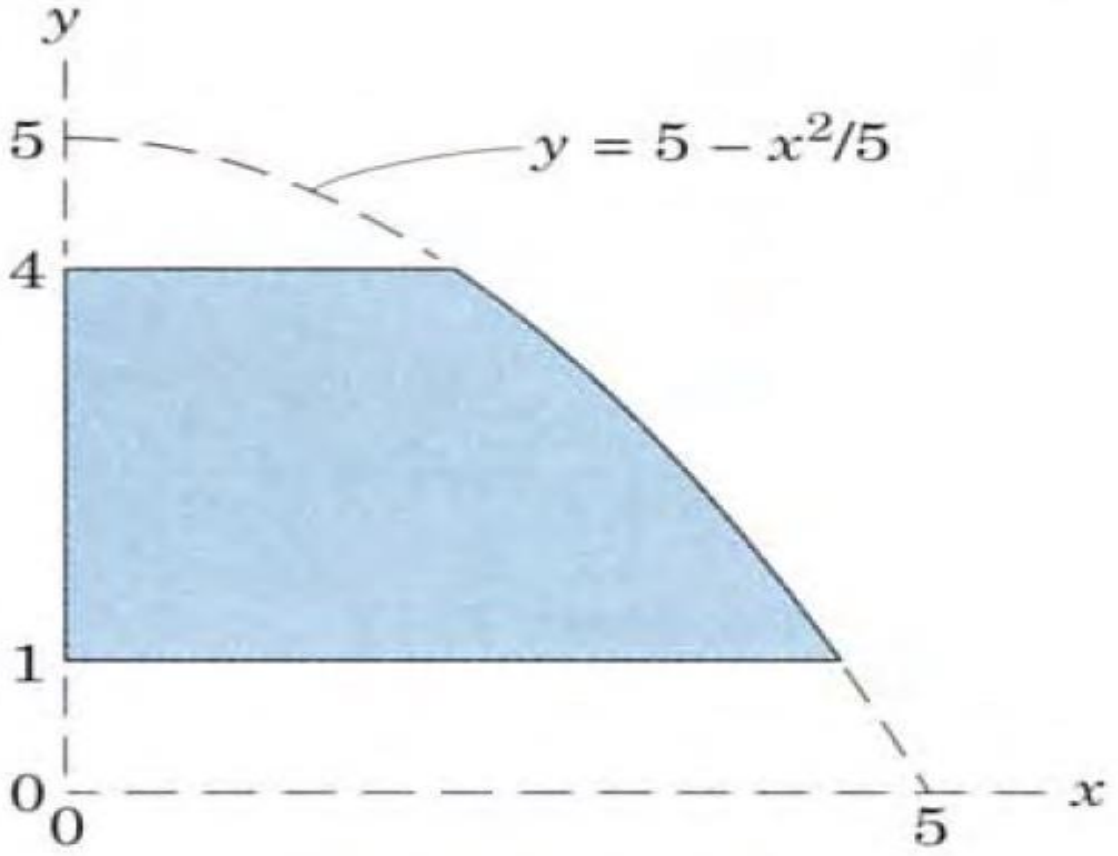
$$\int y_c\,dA = \int_0^a \left(\frac{y_1+y_2}{2}\right)(y_2 - y_1)\,dx = \frac{1}{2}\int_0^a (y_2^2 - y_1^2)\,dx$$
$$= \frac{1}{2}\int_0^a \left(\frac{xb^2}{a} - \frac{x^2b^2}{a^2}\right)dx = \frac{1}{12}ab^2$$

$$\bar{y} = \frac{\int y_c\,dA}{A} = \frac{ab^2/12}{ab/6} = \underline{\frac{b}{2}}$$

5/6 Determine the x- and y-coordinates of the centroid of the shaded area.

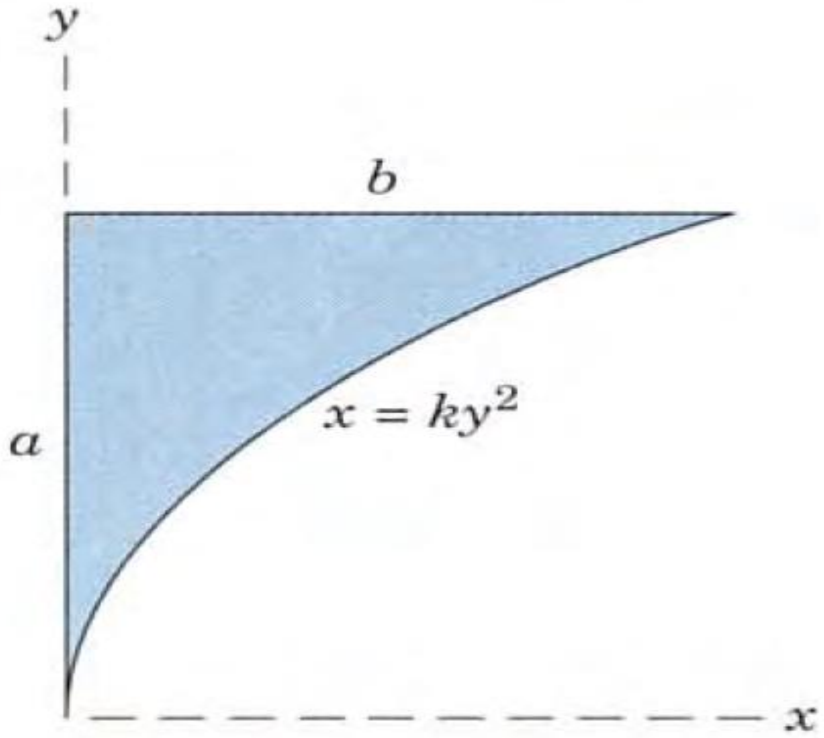


5/8 Locate the centroid of the shaded area shown.



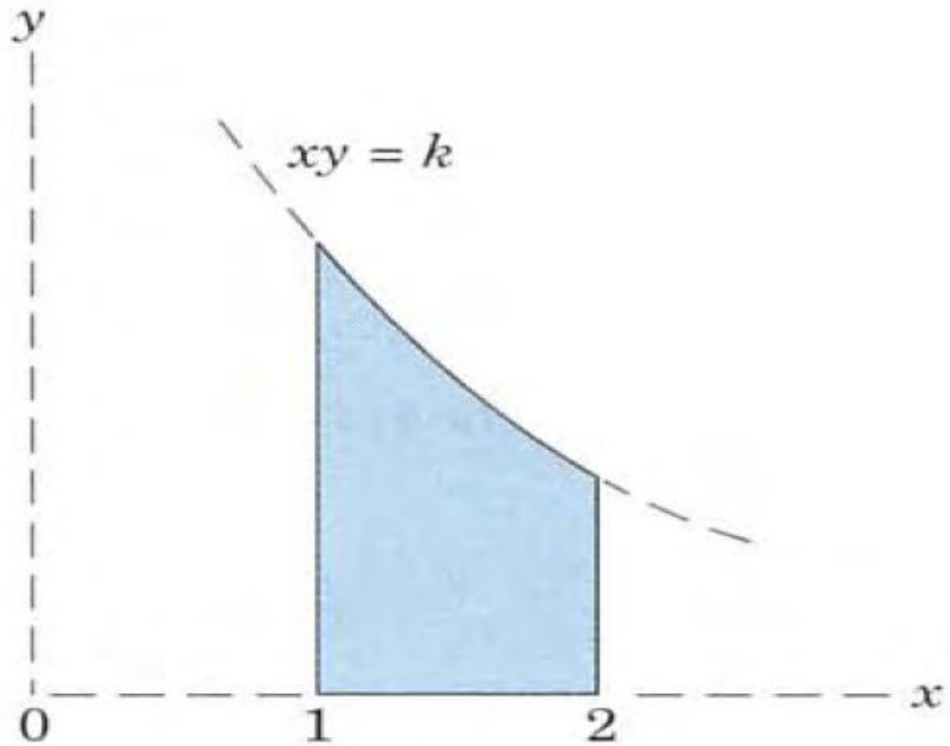
5/9 Determine the coordinates of the centroid of the shaded area.

Ans. $\bar{x} = \frac{3}{10}b$, $\bar{y} = \frac{3}{4}a$



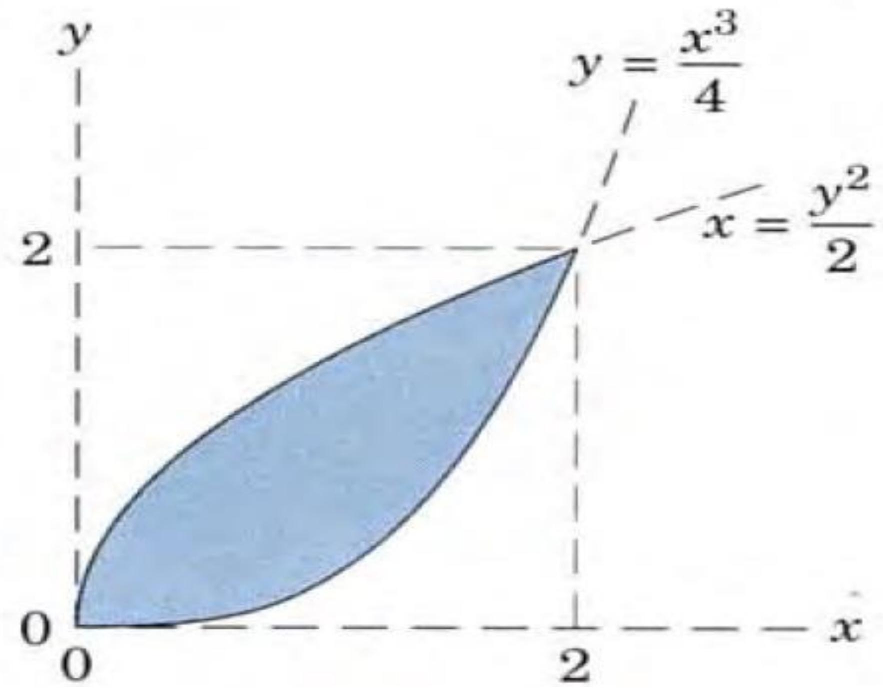
5/11 Determine the coordinates of the centroid of the shaded area.

Ans. $\bar{x} = 1.443$, $\bar{y} = 0.361k$

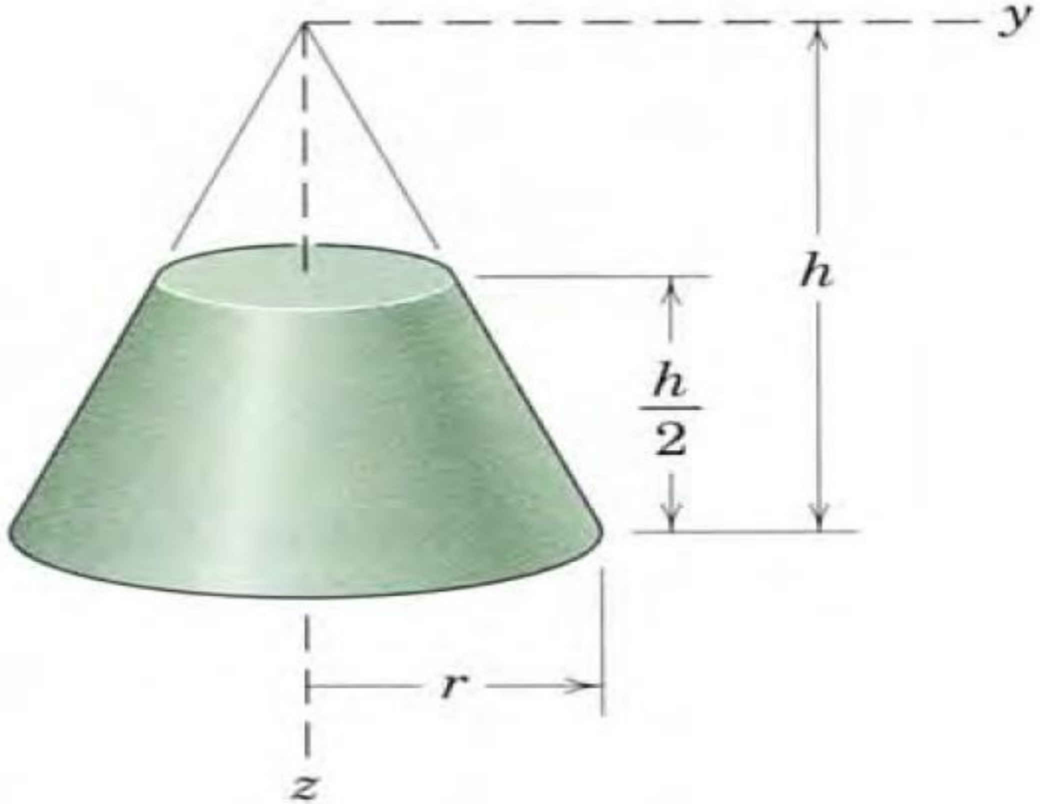


5/23 Locate the centroid of the shaded area between the two curves.

Ans. $\bar{x} = \frac{24}{25}$, $\bar{y} = \frac{6}{7}$

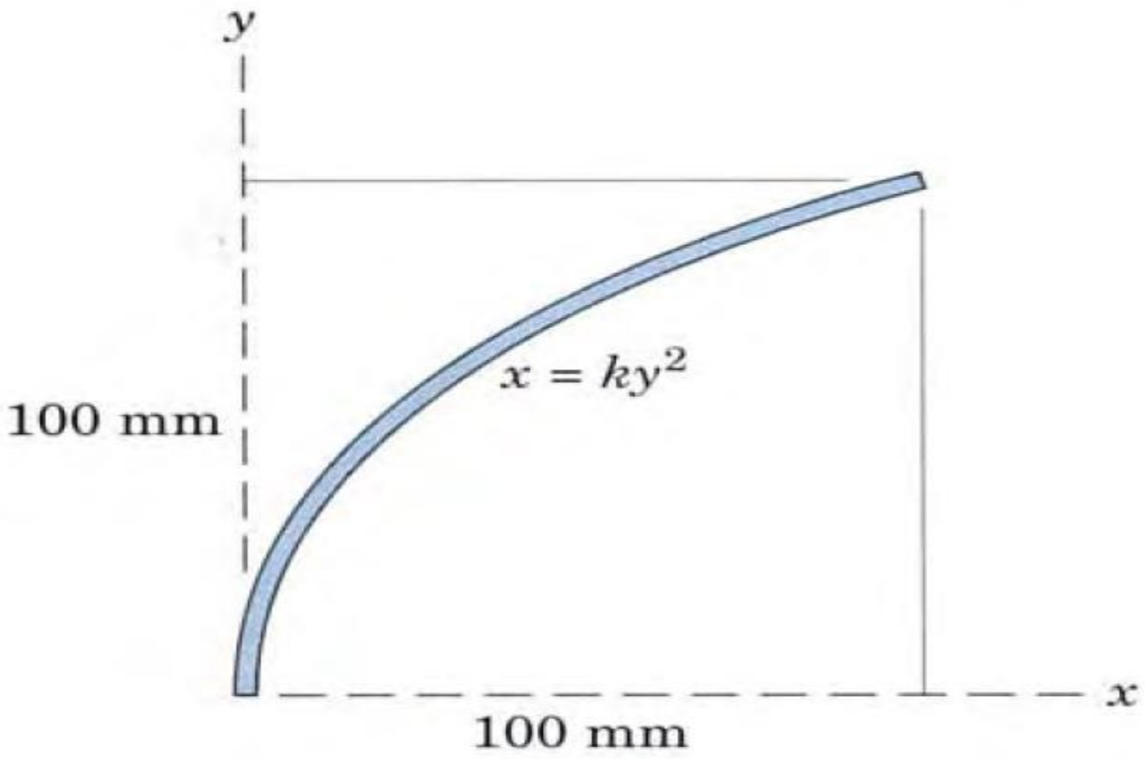


5/30 Calculate the distance $\bar{h}$ measured from the base to the centroid of the volume of the frustum of the right-circular cone.



5/33 The homogeneous slender rod has a uniform cross section and is bent into the shape shown. Calculate the y-coordinate of the mass center of the rod. (*Reminder:* A differential arc length is $dL = \sqrt{(dx)^2 + (dy)^2} = \sqrt{1 + (dx/dy)^2}\,dy$.)

Ans. $\bar{y} = 57.4$ mm



5/24 Determine the coordinates of the centroid of the shaded area.