Moments of Inertia

Introduction

- A *distributed loading* acts perpendicular to an *area* and its *intensity varies* linearly, the computation of the *moment* of the *loading* distribution *about* an *axis* will involve a quantity *called* the *moment of inertia* of the *area* or the *second moment of area*.

Applications

- ❑ Many structural members like beams and columns have cross sectional shapes like I, H, C, etc
- ❑ Why do they usually not have solid rectangular, square, or circular cross sectional areas?
- ❑ What primary property of these members influences design decisions?
- ❑ How can we calculate this property?

- ❑ What parameters of the cross sectional area influence the designer's selection?
- ❑ How can we determine the value of these parameters for a given area?

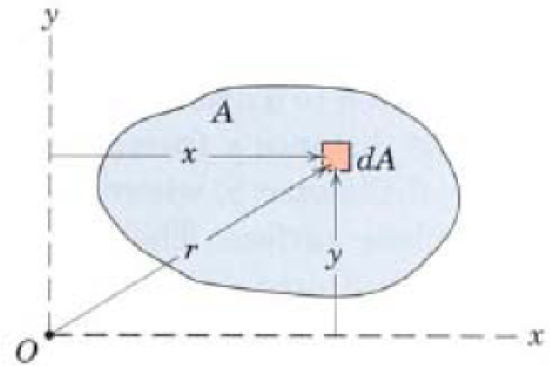
Rectangular and Polar Moments of Inertia

- For the differential area **dA** shown in the figure

- The moments of inertia (**MoI**) of the element **dA** about the **x-** and **y-**axes are

  $dI_x = y^2 dA$ and **$dI_y = x^2 dA$**

- The **MoI** for the entire area **A** about the same axes are obtained by integration



$$I_x = \int y^2 \, dA$$

$$I_y = \int x^2 \, dA$$

Rectangular moments of inertia

Rectangular and Polar Moments of Inertia

- The moment of inertia of **dA** about the pole **O** (z-axis) is

  $dI_z = r^2 dA$

- By integration, the moment of inertia of the entire area about **O** is estimated as

$$I_z = \int r^2 \, dA$$

Polar moments of inertia

- Because $x^2 + y^2 = r^2$, it is clear that

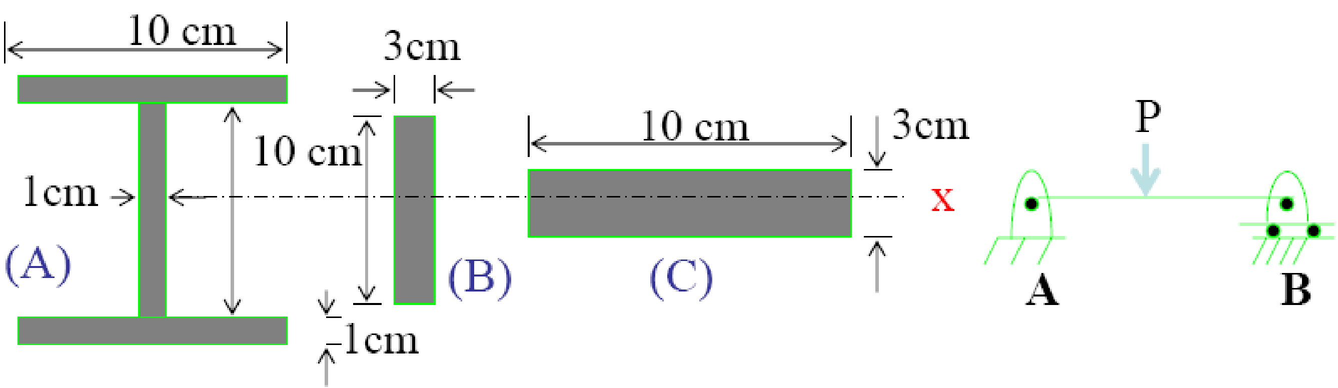
$$I_z = I_x + I_y$$

- The **MoI** is also referred to as the second moment of area and has units of length to the fourth power (**m^4 or in^4**).

Rectangular and Polar Moments of Inertia

- The **MoI** of an element involves the square of the distance from the inertia axis to the element
- Thus an element whose coordinate is negative contributes as much to the moment of inertia as does an equal element with a positive coordinate of the same magnitude
- Consequently the area moment of inertia about any axis is always a positive quantity
- In contrast, the first moment of the area, which was involved in the computations of centroid, could be either positive, negative, or zero
- The choice of the coordinates to use for the calculation of **MoI** is important
- Rectangular coordinates should be used for shapes whose boundaries are most easily expressed in these coordinates
- Polar coordinates will usually simplify problems involving boundaries which are easily described in r and θ

The Importance of MoI for Areas



❑ Consider three different possible cross sectional shapes and areas for the beam **AB**. All have the same total area and, assuming they are made of same material, they will have the same mass per unit length

❑ For the given vertical loading **P** on the beam, which shape will develop less internal stress and deflection? Why?

❑ The answer depends on the **MoI** of the beam about the **x-axis**. It turns out that section A has the highest **MoI** because most of the area is farthest from the **x-axis**. Hence, it has the least stress and deflection

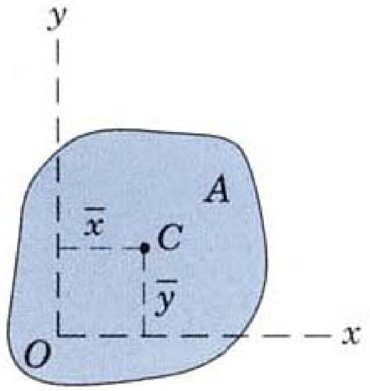
Radius of Gyration of An Area

- Consider an area **A**, as shown in figure a, which has rectangular moments of inertia **Ix**, and I_y, and a polar moment of inertia I_z about **O**

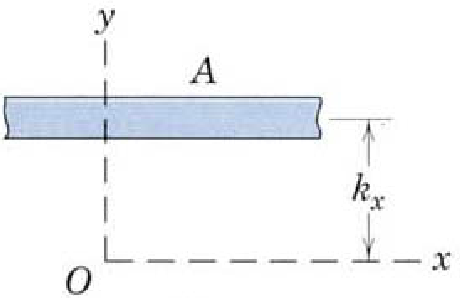
- We now visualize this area as concentrated into a long narrow strip of area **A** a distance **kx** from the **x-axis**, figure b

- By definition the moment of inertia of the strip about the **x-axis** will be the same as that of the original area if $k_x^2 A = I_x$

- The distance k_x is called the ***radius of gyration*** of the area about the **x-axis**



(a)



(b)

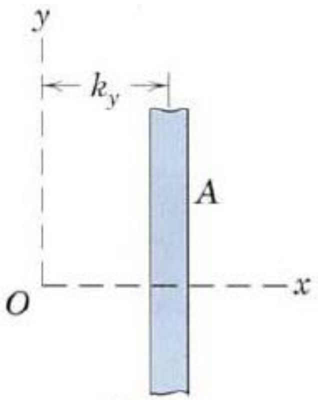
Radius of Gyration of An Area

- A similar relation for the **y-axis** is written by considering the area as concentrated into a narrow strip parallel to the **y-axis** as shown in figure c

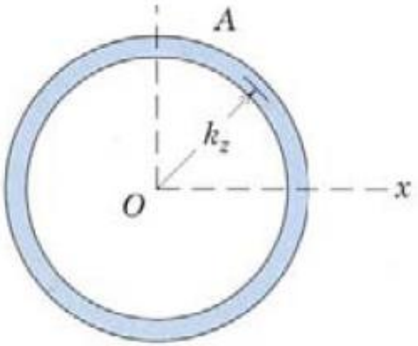
- Also, if we visualize the area as concentrated into a narrow ring of radius $\mathbf{k_z}$ as shown in figure d, we may express the polar moment of inertia as $\mathbf{k_z^2 A = I_z}$

- In summary we write

$$\boxed{\begin{aligned} I_x &= k_x^2 A \\ I_y &= k_y^2 A \\ I_z &= k_z^2 A \end{aligned}} \quad \text{or} \quad \boxed{\begin{aligned} k_x &= \sqrt{I_x/A} \\ k_y &= \sqrt{I_y/A} \\ k_z &= \sqrt{I_z/A} \end{aligned}}$$



(c)



(d)

Radius of Gyration of An Area

- The **radius of gyration** has units of length and gives an indication of the spread of the area from the axes

- This characteristic is important when designing columns

- The **radius of gyration**, then, is a measure of the distribution of the area from the axis in question

- A rectangular or polar moment of inertia may be expressed by specifying the **radius of gyration** and the area

$$\boxed{k_z^2 = k_x^2 + k_y^2}$$

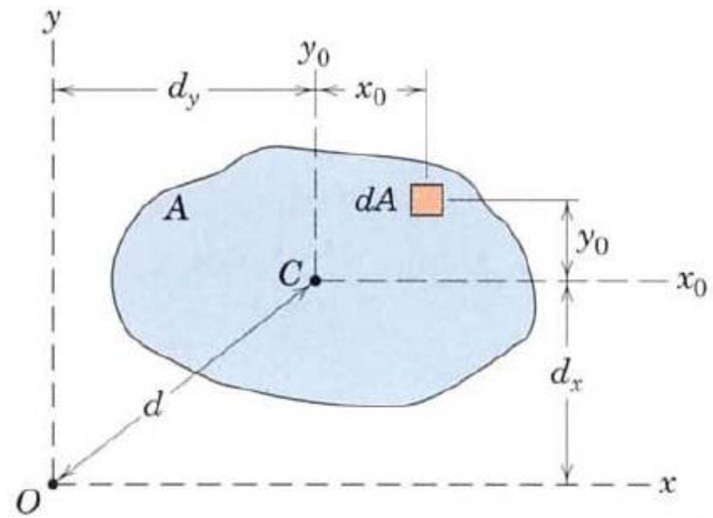
- Thus, the square of the **radius of gyration** about a polar axis equals the sum of the squares of the radii of gyration about the two corresponding rectangular axes

Transfer of Axes

- The **MoI** of an area about a noncentroidal axis may be easily expressed in terms of the **MoI** about a parallel centroidal axis

- In the shown figure, the **x_o-y_o axes** pass through the centroid **C** of the area

- Using the so-called **parallel-axis theorems,** the **MoI** of the area about the **x-y axes** can be expressed as



$$I_x = \bar{I}_x + A{d_x}^2$$
$$I_y = \bar{I}_y + A{d_y}^2$$

$$I_z = \bar{I}_z + Ad^2$$

Parallel-axis Theorems

- Two points in particular should be noted. First, the axes between which the transfer is made must be parallel, and second, one of the axes must pass through the centroid of the area

- If a transfer is desired between two parallel axes neither of which passes through the centroid, it is first necessary to transfer from one axis to the parallel centroidal axis and then to transfer from the centroidal axis to the second axis

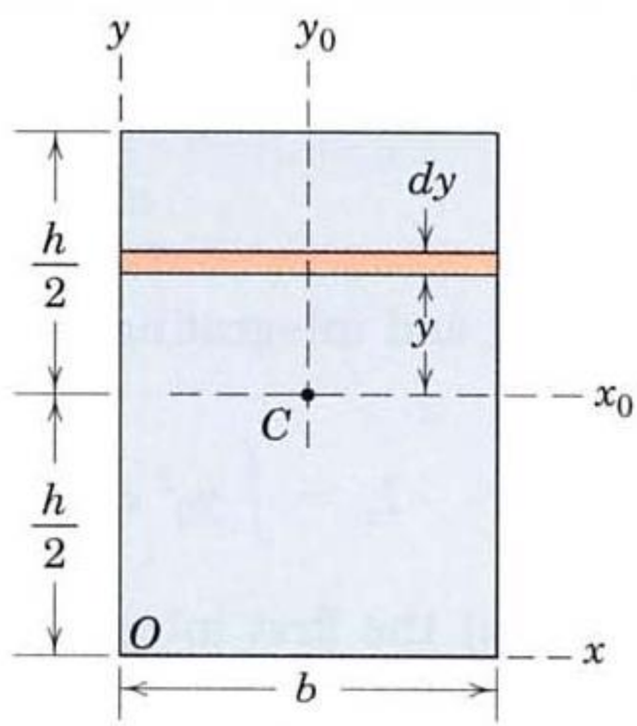
- The parallel-axis theorems also hold for radii of gyration as follows:

$$\boxed{k^2 = \bar{k}^2 + d^2}$$

- where $\bar{k}$ is the radius of gyration about a centroidal axis parallel to the axis about which k applies and d is the distance between the two axes

Example

Determine the moments of inertia of the rectangular area about the centroidal x_0- and y_0-axes, the centroidal polar axis z_0 through C, the x-axis, and the polar axis z through O.



Solution

Solution. For the calculation of the moment of inertia $\bar{I}_x$ about the x_0-axis, a horizontal strip of area $b\,dy$ is chosen so that all elements of the strip have the same y-coordinate. Thus,

$$\left[I_x = \int y^2\,dA\right] \qquad \bar{I}_x = \int_{-h/2}^{h/2} y^2 b\,dy = \tfrac{1}{12}bh^3 \qquad \textit{Ans.}$$

By interchange of symbols, the moment of inertia about the centroidal y_0-axis is

$$\bar{I}_y = \tfrac{1}{12}hb^3 \qquad \textit{Ans.}$$

The centroidal polar moment of inertia is

$$[\bar{I}_z = \bar{I}_x + \bar{I}_y] \qquad \bar{I}_z = \tfrac{1}{12}(bh^3 + hb^3) = \tfrac{1}{12}A(b^2 + h^2) \qquad \textit{Ans.}$$

By the parallel-axis theorem the moment of inertia about the x-axis is

$$[I_x = \bar{I}_x + A{d_x}^2] \qquad I_x = \tfrac{1}{12}bh^3 + bh\left(\frac{h}{2}\right)^2 = \tfrac{1}{3}bh^3 = \tfrac{1}{3}Ah^2 \qquad \textit{Ans.}$$

Solution

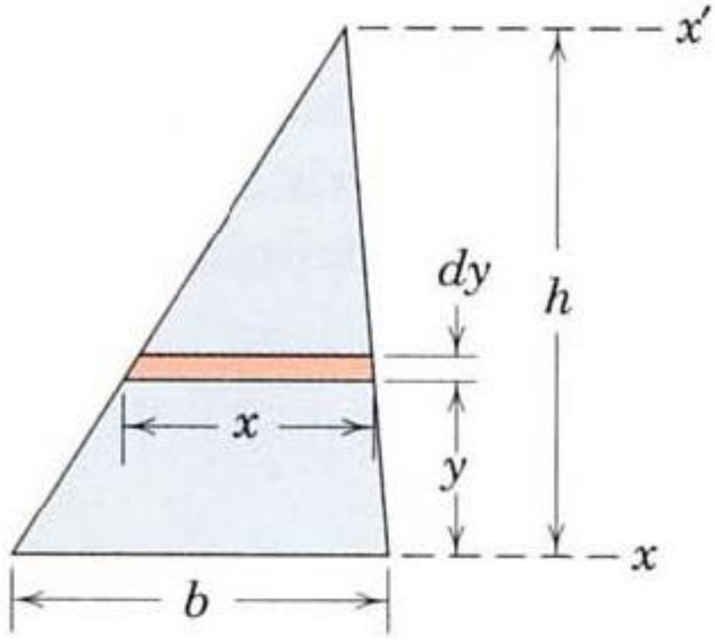
We also obtain the polar moment of inertia about O by the parallel-axis theorem, which gives us

$$[I_z = \bar{I}_z + Ad^2] \qquad I_z = \tfrac{1}{12}A(b^2 + h^2) + A\left[\left(\frac{b}{2}\right)^2 + \left(\frac{h}{2}\right)^2\right]$$

$$I_z = \tfrac{1}{3}A(b^2 + h^2) \qquad \textit{Ans.}$$

Example

Determine the moments of inertia of the triangular area about its base and about parallel axes through its centroid and vertex.



Solution

Solution. A strip of area parallel to the base is selected as shown in the figure, and it has the area $dA = x\,dy = [(h - y)b/h]\,dy$. By definition

$$[I_x = \int y^2\,dA] \qquad I_x = \int_0^h y^2 \frac{h - y}{h} b\,dy = b\left[\frac{y^3}{3} - \frac{y^4}{4h}\right]_0^h = \frac{bh^3}{12} \qquad \textit{Ans.}$$

By the parallel-axis theorem the moment of inertia $\bar{I}$ about an axis through the centroid, a distance $h/3$ above the x-axis, is

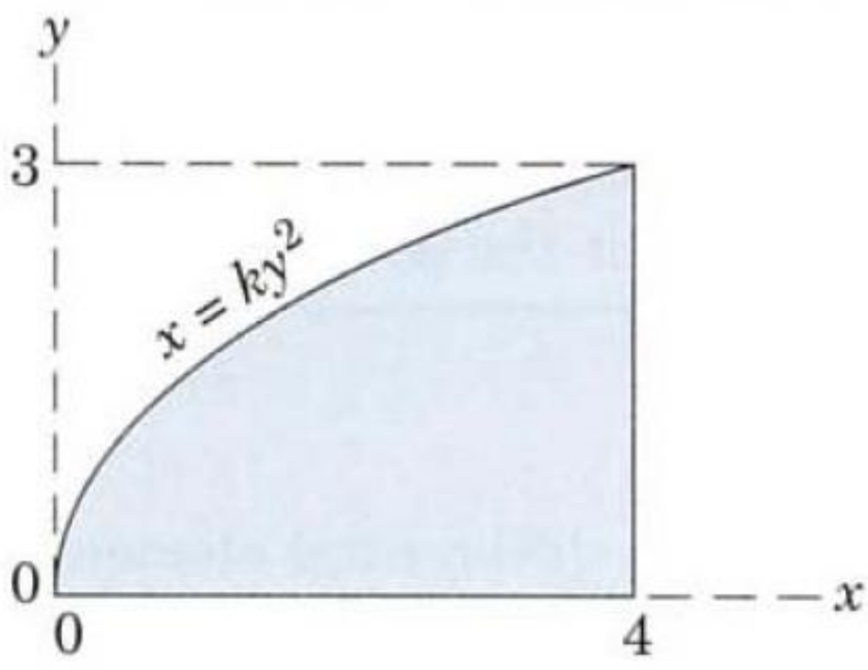
$$[\bar{I} = I - Ad^2] \qquad \bar{I} = \frac{bh^3}{12} - \left(\frac{bh}{2}\right)\left(\frac{h}{3}\right)^2 = \frac{bh^3}{36} \qquad \textit{Ans.}$$

A transfer from the centroidal axis to the x'-axis through the vertex gives

$$[I = \bar{I} + Ad^2] \qquad I_{x'} = \frac{bh^3}{36} + \left(\frac{bh}{2}\right)\left(\frac{2h}{3}\right)^2 = \frac{bh^3}{4} \qquad \textit{Ans.}$$

Example

Determine the moment of inertia of the area under the parabola about the x-axis. Solve by using (a) a horizontal strip of area and (b) a vertical strip of area.

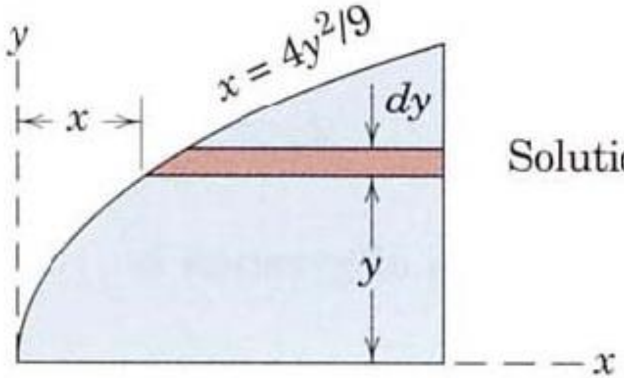


Solution

Solution. The constant $k = \frac{4}{9}$ is obtained first by substituting $x = 4$ and $y = 3$ into the equation for the parabola.

(a) Horizontal strip. Since all parts of the horizontal strip are the same distance from the x-axis, the moment of inertia of the strip about the x-axis is $y^2\, dA$ where $dA = (4 - x)\, dy = 4(1 - y^2/9)\, dy$. Integrating with respect to y gives us

$$\left[I_x = \int y^2\, dA\right] \quad I_x = \int_0^3 4y^2 \left(1 - \frac{y^2}{9}\right) dy = \frac{72}{5} = 14.40 \text{ (units)}^4 \qquad \textit{Ans.}$$



Solution (a)

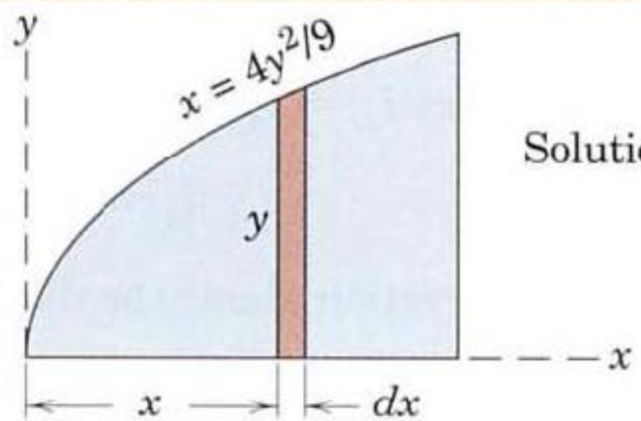
Solution

(b) Vertical strip. Here all parts of the element are at different distances from the x-axis, so we must use the correct expression for the moment of inertia of the elemental rectangle about its base, which, from Sample Problem A/1, is $bh^3/3$. For the width dx and the height y the expression becomes

$$dI_x = \tfrac{1}{3}(dx)y^3$$

To integrate with respect to x, we must express y in terms of x, which gives $y = 3\sqrt{x}/2$, and the integral becomes

$$I_x = \tfrac{1}{3}\int_0^4 \left(\frac{3\sqrt{x}}{2}\right)^3 dx = \frac{72}{5} = 14.40 \text{ (units)}^4 \qquad \textit{Ans.}$$

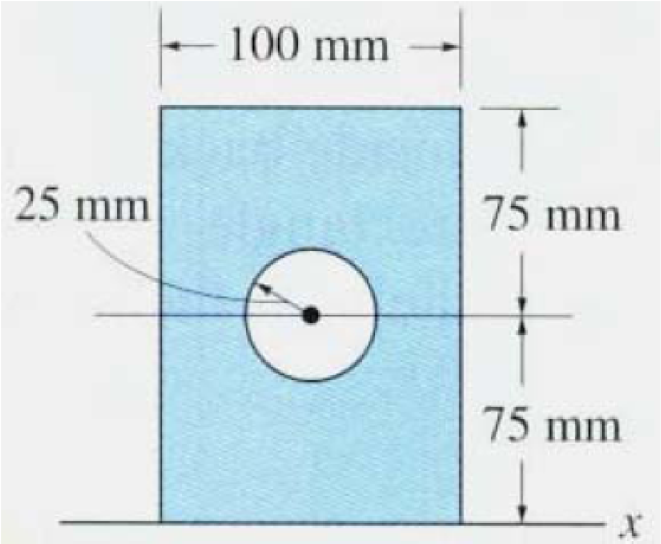


Solution (b)

Area Moments of Inertia
Composite Areas

- It is frequently necessary to calculate the moment of inertia of an area composed of a number of distinct parts of simple and calculable geometric shape

- Because a moment of inertia is the integral or sum of the products of distance squared times element of area, it follows that the moment of inertia of a positive area is always a positive quantity



Area Moments of Inertia
Composite Areas

- ❑ The moment of inertia of a composite area about a particular axis is therefore simply the sum of the moments of inertia of its component parts about the same axis

- ❑ It is often convenient to regard a composite area as being composed of positive and negative parts

- ❑ We may then treat the moment of inertia of a negative area as a negative quantity

Composite Areas
Steps for Analysis

1. Divide the given area into its simpler shaped parts
2. Locate the centroid of each "simpler" shaped part and indicate the perpendicular distance from each centroid to the desired reference axis
3. The MoI of these "simpler" shaped areas about their centroidal axes are found inside back cover of the textbook
4. Determine the MoI of each "simpler" shaped part about the desired reference axis using the parallel-axis theorem
5. When a composite area is composed of a large number of parts, it is convenient to tabulate the results for each of the parts in terms of its area A, its centroidal moment of inertia I, the distance d from its centroidal axis to the axis about which the moment of inertia of the entire section is being computed, and the product Ad^2.

Composite Areas
Steps for Analysis

6. The MoI of the entire area about the reference axis is determined by performing an algebraic summation of the individual MoIs obtained in Step 4. (Please note that MoI of a hole is subtracted)

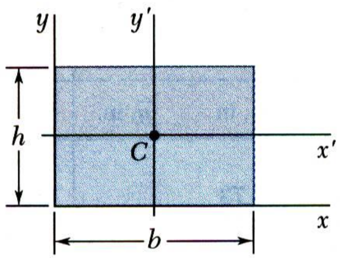
Part	Area, A	d_x	d_y	$Ad_x^{\,2}$	$Ad_y^{\,2}$	$\bar{I}_x$	$\bar{I}_y$
Sums	ΣA			$\Sigma Ad_x^{\,2}$	$\Sigma Ad_y^{\,2}$	$\Sigma \bar{I}_x$	$\Sigma \bar{I}_y$

7. From the sums of the four columns, then, the moments of inertia for the composite area about the x- and y-axes become

$$I_x = \Sigma \bar{I}_x + \Sigma A d_x^{\,2}$$
$$I_y = \Sigma \bar{I}_y + \Sigma A d_y^{\,2}$$

Moments of Inertia of Composite Areas

Rectangle



$\bar{I}_{x'} = \frac{1}{12} bh^3$

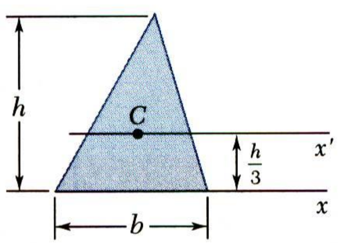
$\bar{I}_{y'} = \frac{1}{12} b^3 h$

$I_x = \frac{1}{3} bh^3$

$I_y = \frac{1}{3} b^3 h$

$J_C = \frac{1}{12} bh(b^2 + h^2)$

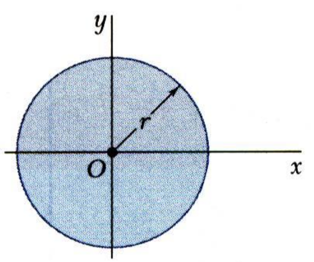
Triangle



$\bar{I}_{x'} = \frac{1}{36} bh^3$

$I_x = \frac{1}{12} bh^3$

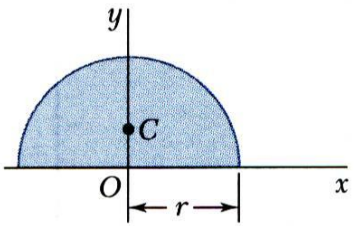
Circle



$\bar{I}_x = \bar{I}_y = \frac{1}{4}\pi r^4$

$J_O = \frac{1}{2}\pi r^4$

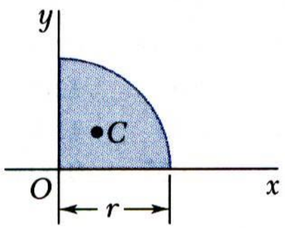
Semicircle



$I_x = I_y = \frac{1}{8}\pi r^4$

$J_O = \frac{1}{4}\pi r^4$

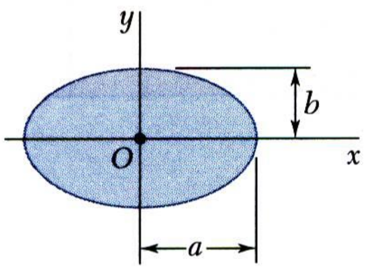
Quarter circle



$I_x = I_y = \frac{1}{16}\pi r^4$

$J_O = \frac{1}{8}\pi r^4$

Ellipse



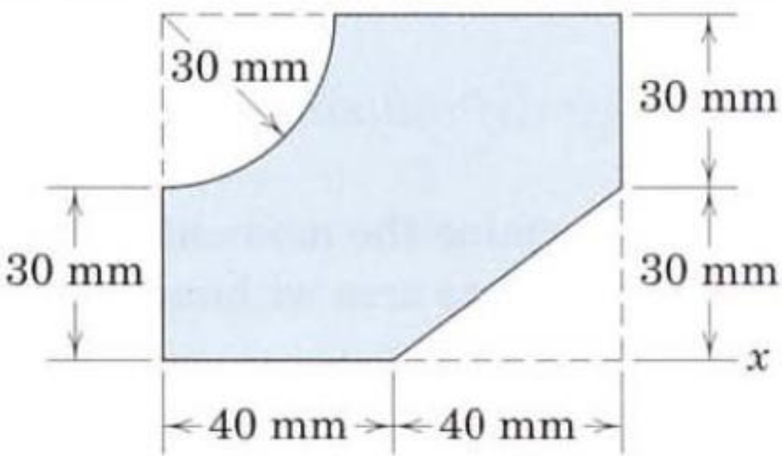
$\bar{I}_x = \frac{1}{4}\pi ab^3$

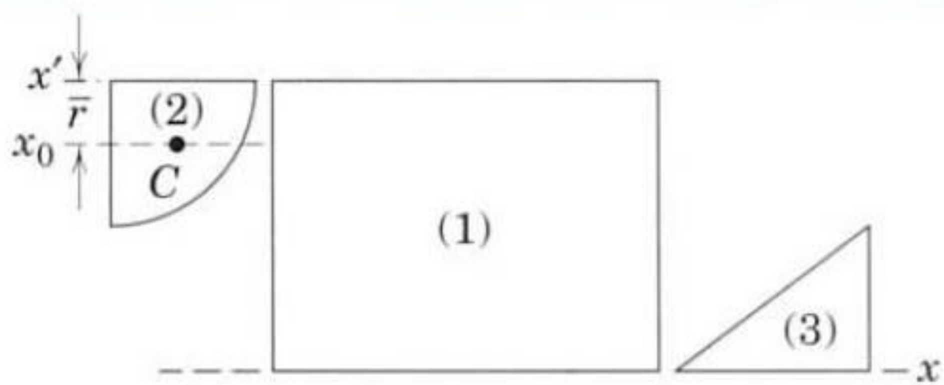
$\bar{I}_y = \frac{1}{4}\pi a^3 b$

$J_O = \frac{1}{4}\pi ab(a^2 + b^2)$

Example

Calculate the moment of inertia and radius of gyration about the x-axis for the shaded area shown.





Solution

Solution. The composite area is composed of the positive area of the rectangle (1) and the negative areas of the quarter circle (2) and triangle (3). For the rectangle the moment of inertia about the x-axis, from Sample Problem A/1 (or Table D/3), is

$$I_x = \tfrac{1}{3}Ah^2 = \tfrac{1}{3}(80)(60)(60)^2 = 5.76(10^6)\text{ mm}^4$$

From Sample Problem A/3 (or Table D/3), the moment of inertia of the negative quarter-circular area about its base axis x' is

$$I_{x'} = -\frac{1}{4}\left(\frac{\pi r^4}{4}\right) = -\frac{\pi}{16}(30)^4 = -0.1590(10^6)\text{ mm}^4$$

We now transfer this result through the distance $\bar{r} = 4r/(3\pi) = 4(30)/(3\pi) = 12.73$ mm by the transfer-of-axis theorem to get the centroidal moment of inertia of part (2) (or use Table D/3 directly).

$$[\bar{I} = I - Ad^2] \qquad \bar{I}_x = -0.1590(10^6) - \left[-\frac{\pi(30)^2}{4}(12.73)^2\right]$$

$$= -0.0445(10^6)\text{ mm}^4$$

Solution

The moment of inertia of the quarter-circular part about the x-axis is now

$$[I = \bar{I} + Ad^2] \qquad I_x = -0.0445(10^6) + \left[-\frac{\pi(30)^2}{4}\right](60 - 12.73)^2$$

$$= -1.624(10^6)\ \text{mm}^4$$

Finally, the moment of inertia of the negative triangular area (3) about its base, from Sample Problem A/2 (or Table D/3), is

$$I_x = -\tfrac{1}{12}bh^3 = -\tfrac{1}{12}(40)(30)^3 = -0.09(10^6)\ \text{mm}^4$$

The total moment of inertia about the x-axis of the composite area is, consequently,

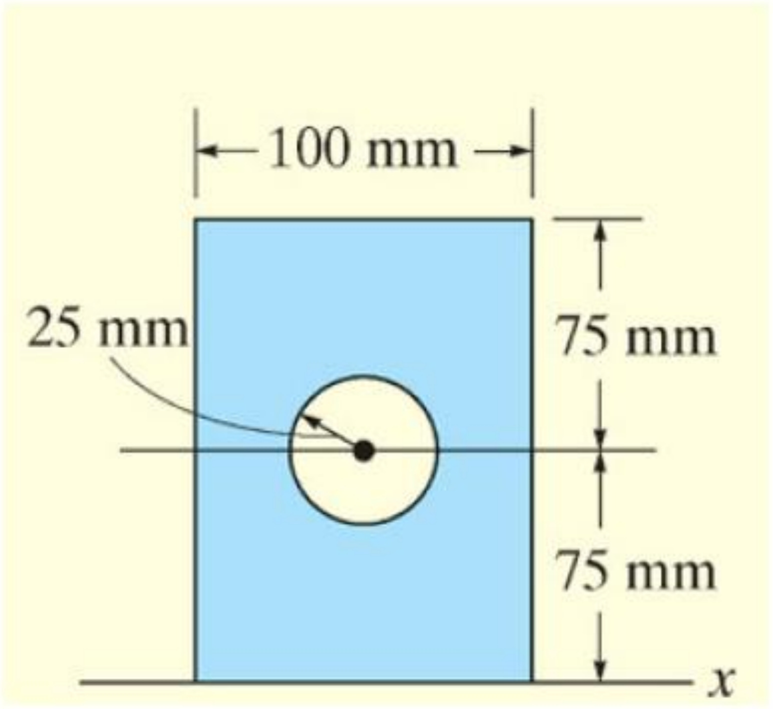
$$I_x = 5.76(10^6) - 1.624(10^6) - 0.09(10^6) = 4.05(10^6)\ \text{mm}^4 \qquad \textit{Ans.}$$

The net area of the figure is $A = 60(80) - \frac{1}{4}\pi(30)^2 - \frac{1}{2}(40)(30) = 3490\ \text{mm}^2$ so that the radius of gyration about the x-axis is

$$k_x = \sqrt{I_x/A} = \sqrt{4.05(10^6)/3490} = 34.0\ \text{mm} \qquad \textit{Ans.}$$

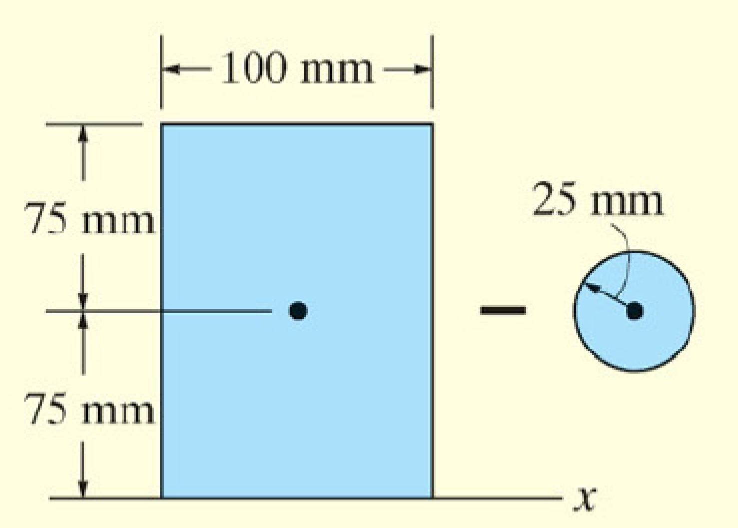
Example

Compute the moment of inertia for the composite area in the shown figure



Solution

Composite Parts. The composite area is obtained by *subtracting* the circle from the rectangle as shown in the figure. The centroid of each area is located in the figure



Parallel-Axis Theorem. The moments of inertia about the x axis are determined using the parallel-axis theorem and the data in the table on the inside back cover

Circle:

$$I_x = \bar{I}_{x'} + Ad_y^2$$

$$= \frac{1}{4}\pi(25)^4 + \pi(25)^2(75)^2 = 11.4(10^6)\ \text{mm}^4$$

Solution

Rectangle:

$$I_x = \bar{I}_{x'} + Ad_y^2$$

$$= \frac{1}{12}(100)(150)^3 + (100)(150)(75)^2 = 112.5(10^6)\ \text{mm}^4$$

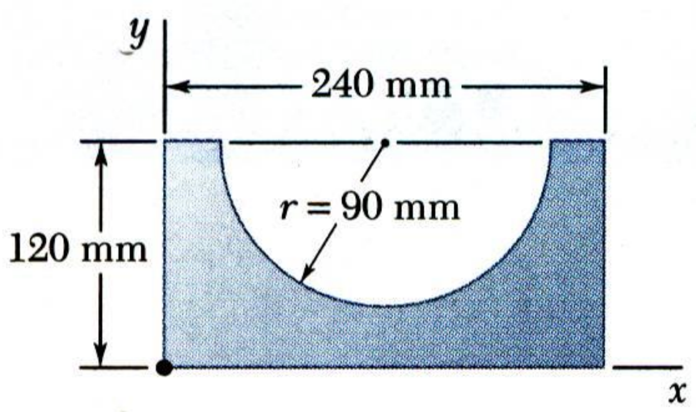
Summation:

The moment of inertia for the composite area is thus

$$I_x = -11.4(10^6) + 112.5(10^6)$$

$$= 101(10^6)\ \text{mm}^4$$

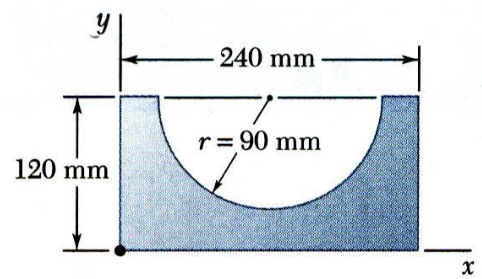
Example



Determine the moment of inertia of the shaded area with respect to the *x* axis.

SOLUTION:

- Compute the moments of inertia of the bounding rectangle and half-circle with respect to the *x* axis.
- The moment of inertia of the shaded area is obtained by subtracting the moment of inertia of the half-circle from the moment of inertia of the rectangle.

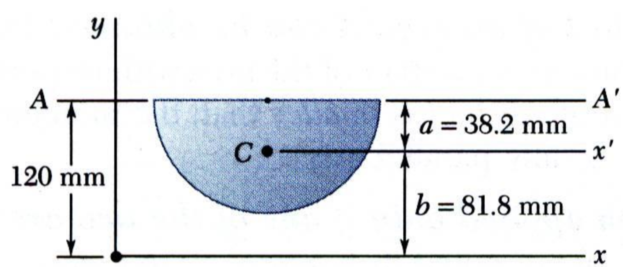


SOLUTION:

- Compute the moments of inertia of the bounding rectangle and half-circle with respect to the *x* axis.

Rectangle:

$$I_x = \tfrac{1}{3}bh^3 = \tfrac{1}{3}(240)(120) = 138.2\times10^6\,\mathrm{mm}^4$$



$$a = \frac{4r}{3\pi} = \frac{(4)(90)}{3\pi} = 38.2\ \mathrm{mm}$$

$$\mathrm{b} = 120 \text{ - } \mathrm{a} = 81.8\ \mathrm{mm}$$

$$A = \tfrac{1}{2}\pi r^2 = \tfrac{1}{2}\pi(90)^2$$

$$= 12.72\times10^3\,\mathrm{mm}^2$$

Half-circle:

moment of inertia with respect to *AA'*,

$$I_{AA'} = \tfrac{1}{8}\pi r^4 = \tfrac{1}{8}\pi(90)^4 = 25.76\times10^6\,\mathrm{mm}^4$$

moment of inertia with respect to *x'*,

$$\bar{I}_{x'} = I_{AA'} - Aa^2 = \left(25.76\times10^6\right) - \left(12.72\times10^3\right)\left(38.2^2\right)$$

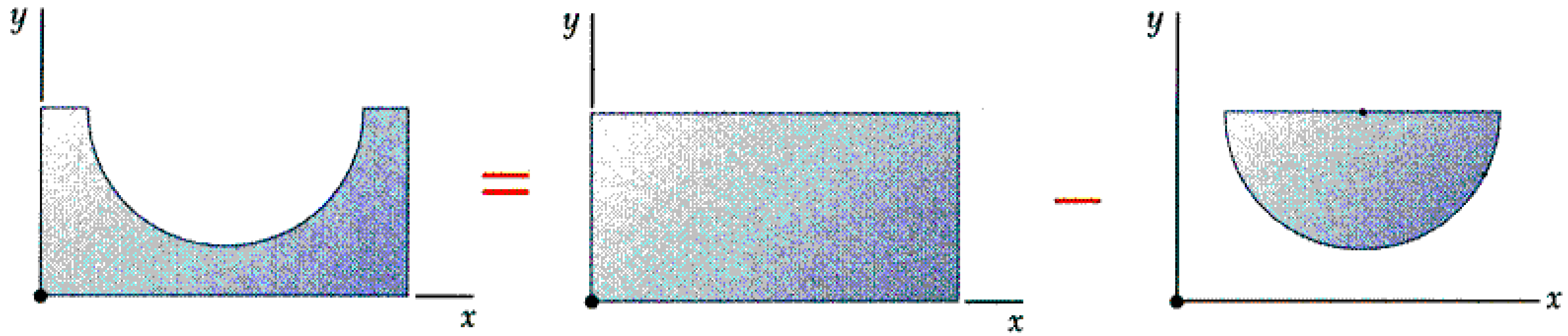
$$= 7.20\times10^6\,\mathrm{mm}^4$$

moment of inertia with respect to *x*,

$$I_x = \bar{I}_{x'} + Ab^2 = 7.20\times10^6 + \left(12.72\times10^3\right)(81.8)^2$$

$$= 92.3\times10^6\,\mathrm{mm}^4$$

- The moment of inertia of the shaded area is obtained by subtracting the moment of inertia of the half-circle from the moment of inertia of the rectangle.



$$I_x \quad = \quad 138.2\times10^6\,\text{mm}^4 \quad - \quad 92.3\times10^6\,\text{mm}^4$$

$$\boxed{I_x = 45.9\times10^6\,\text{mm}^4}$$