CHAPTER 2
SECTION B.

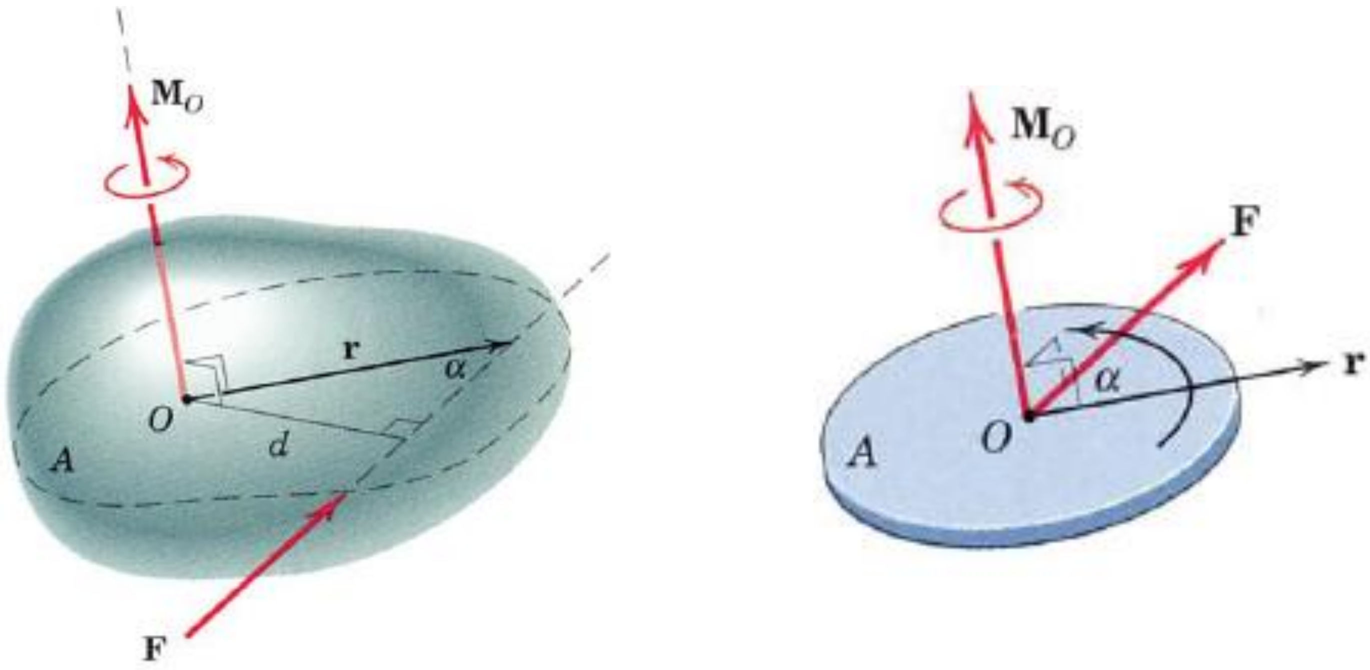
Three-Dimensional Force Systems

2/8 Moment and Couple

In two dimensional analyses it is often convenient to determine a moment magnitude by scalar multiplication using the moment-arm rule.

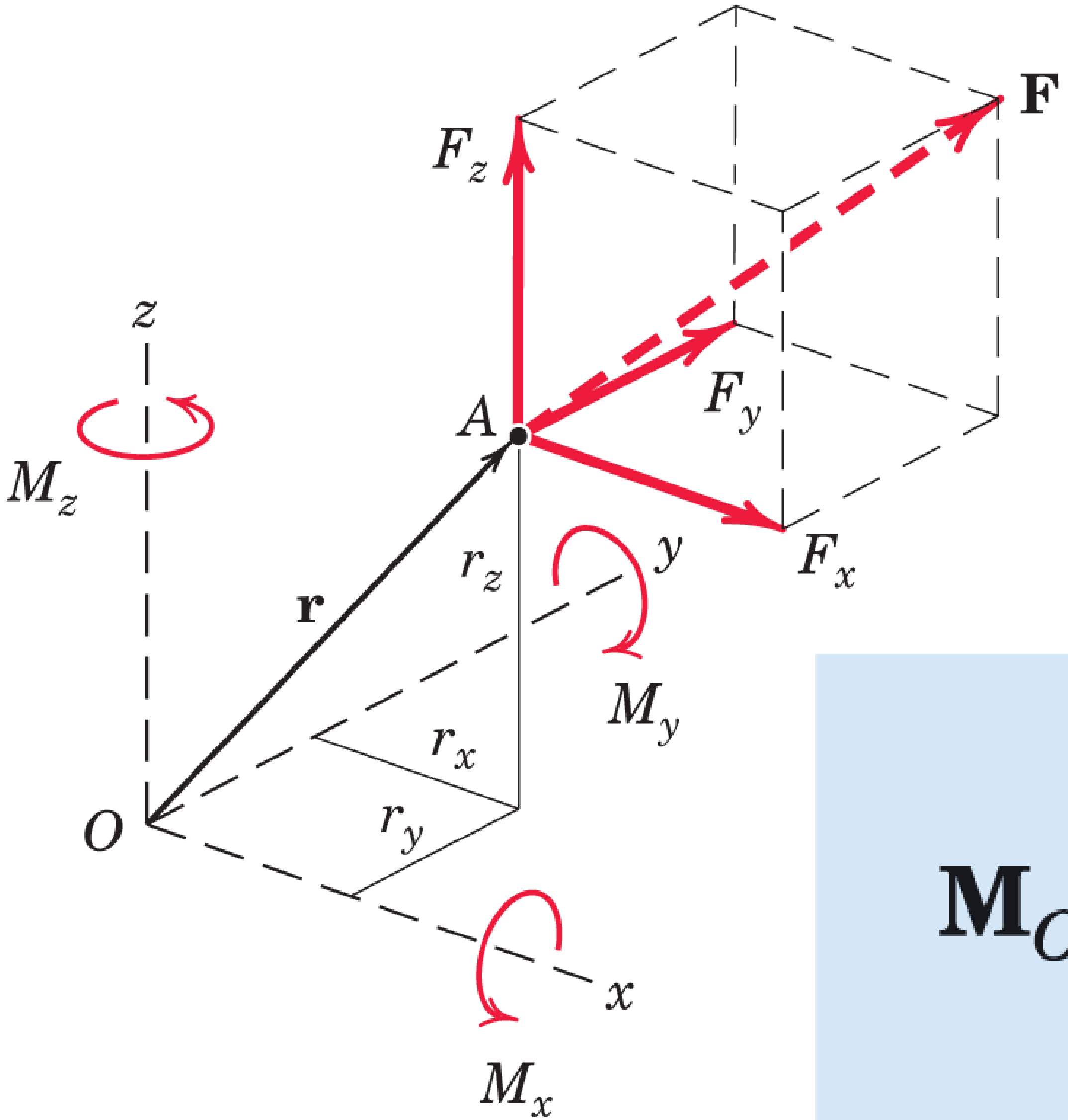
In three dimensions, however, the determination of the perpendicular distance between a point or line and the line of action of the force can be a tedious computation. A vector approach with cross-product multiplication then becomes advantages.

Moment in 3-D Vector Formulation



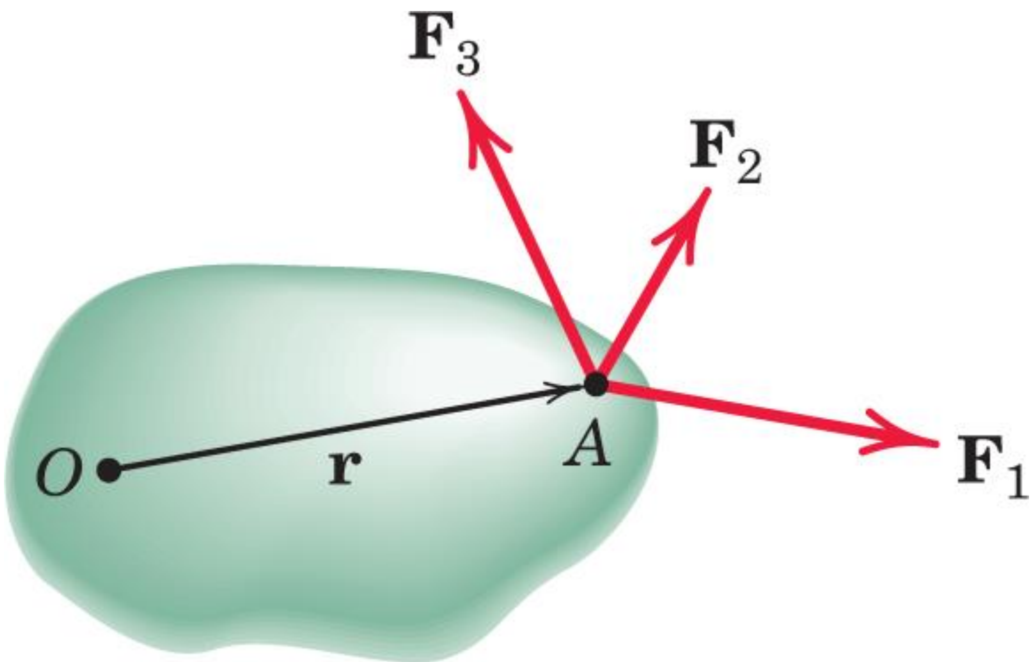
Using the vector cross product, $\mathbf{M_O} = \mathbf{r} \times \mathbf{F}$.

Here **r** is the position vector from point O to any point on the line of action of ***F***.



$$\mathbf{M}_O = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ r_x & r_y & r_z \\ F_x & F_y & F_z \end{vmatrix}$$

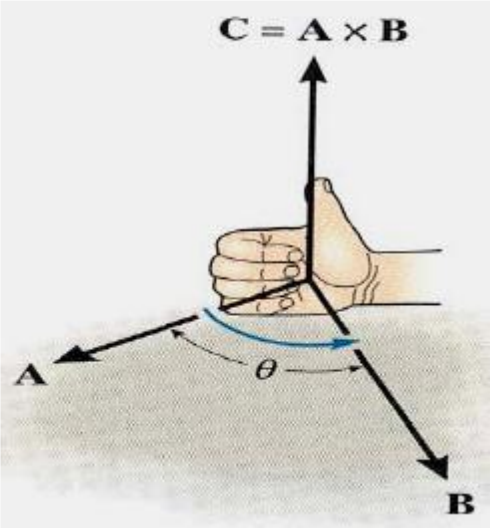
Varignon's Theorem in 3-D

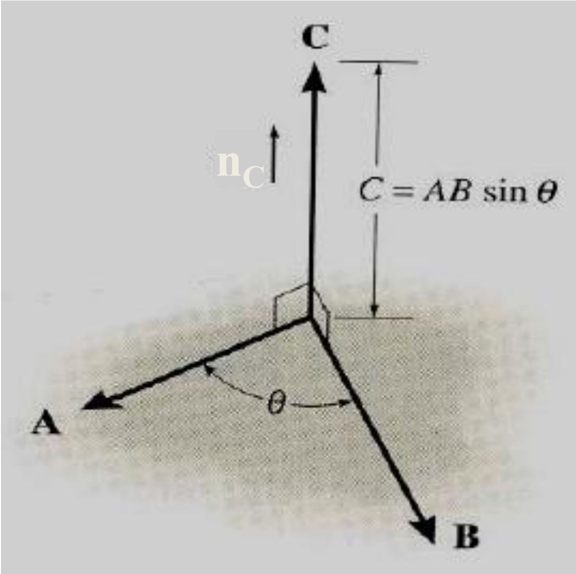


$$\vec{r}\times\vec{F}_1+\vec{r}\times\vec{F}_2+\ldots+\vec{r}\times\vec{F}_n=\vec{r}\times\left(\vec{F}_1+\vec{F}_2+\ldots+\vec{F}_n\right)$$

$$\mathbf{M}_O = \Sigma(\mathbf{r}\times\mathbf{F}) = \mathbf{r}\times\mathbf{R}$$

Cross Product





In general, the cross product of two vectors **A** and **B** results in another vector **C** , i.e., **C = A** ×**B**. The magnitude and direction of the resulting vector can be written as

$\mathbf{C} = \mathbf{A} \times \mathbf{B} = A\, B \sin\theta\, \mathbf{n}_C$

Laws of operation

special commutative law	$\mathbf{A} \times \mathbf{B} = -\,\mathbf{B} \times \mathbf{A}$
scalar multiplication	$a\,(\mathbf{A} \times \mathbf{B}) = (a\,\mathbf{A}) \times \mathbf{B} = \mathbf{A} \times (a\,\mathbf{B})$
distributive law	$\mathbf{A} \times (\mathbf{B} + \mathbf{C}) = \mathbf{A} \times \mathbf{B} + \mathbf{A} \times \mathbf{C}$

1.Cross product between Cartesian vectors

Let $\mathbf{A} = A_x\,\mathbf{i} + A_y\,\mathbf{j} + A_z\,\mathbf{k}$ and $\mathbf{B} = B_x\,\mathbf{i} + B_y\,\mathbf{j} + B_z\,\mathbf{k}$

Then $\mathrm{A} \times \mathrm{B} =$

$$(A_y B_z - A_z B_y)\,\mathbf{i} - (A_x B_z - A_z B_x)\,\mathbf{j} + (A_x B_y - A_y B_x)\,\mathbf{k}$$

more ...

2.computation of cross product by expansion of determinant

$$\mathbf{A}\times\mathbf{B} = \det\begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{bmatrix} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix} \begin{matrix} \mathbf{i} & \mathbf{j} \\ A_x & A_y \\ B_x & B_y \end{matrix}$$

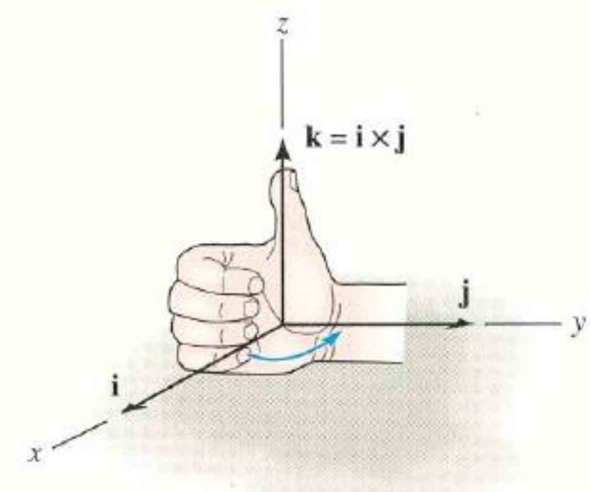
$$\mathbf{A}\times\mathbf{B} = A_y B_z \mathbf{i} + A_z B_x \mathbf{j} + A_x B_y \mathbf{k} - (A_y B_x \mathbf{k} + A_z B_y \mathbf{i} + A_x B_z \mathbf{j})$$

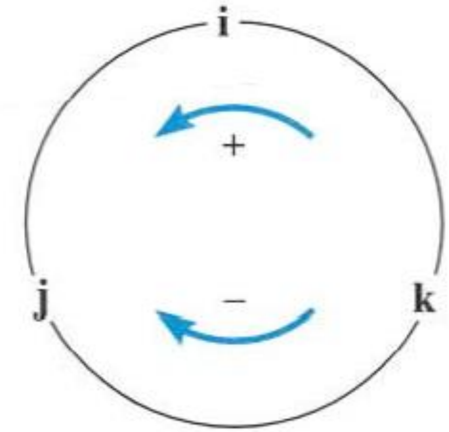
3.expand determinat using co-factors

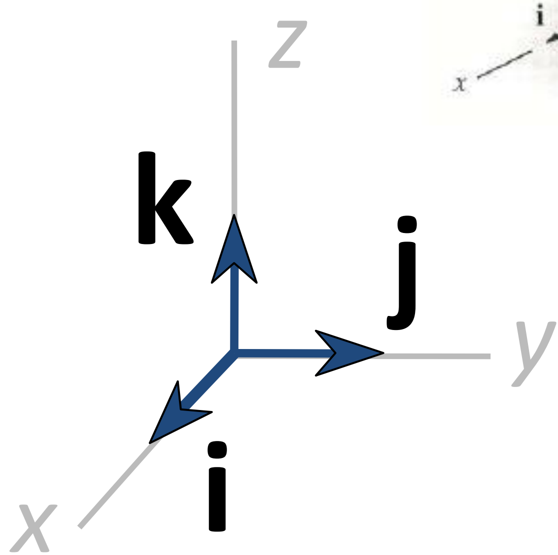
$$\mathbf{A}\times\mathbf{B} = \det\begin{bmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ A_x & A_y & A_z\\ B_x & B_y & B_z\end{bmatrix}$$

$$= \mathbf{i}\begin{vmatrix}\cdot & \cdot & \cdot\\ \cdot & A_y & A_z\\ \cdot & B_y & B_z\end{vmatrix} - \mathbf{j}\begin{vmatrix}\cdot & \cdot & \cdot\\ A_x & \cdot & A_z\\ B_x & \cdot & B_z\end{vmatrix} + \mathbf{k}\begin{vmatrix}\cdot & \cdot & \cdot\\ A_x & A_y & \cdot\\ B_x & B_y & \cdot\end{vmatrix}$$

$$= \mathbf{i}(A_yB_z - B_yA_z) - \mathbf{j}(A_xB_z - B_xA_z) + \mathbf{k}(A_xB_y - B_xA_y)$$







Let **A** = **i** and **B** = **j**. Then

$$\mathbf{i} \times \mathbf{j} = \det \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} = \mathbf{k} \quad !$$

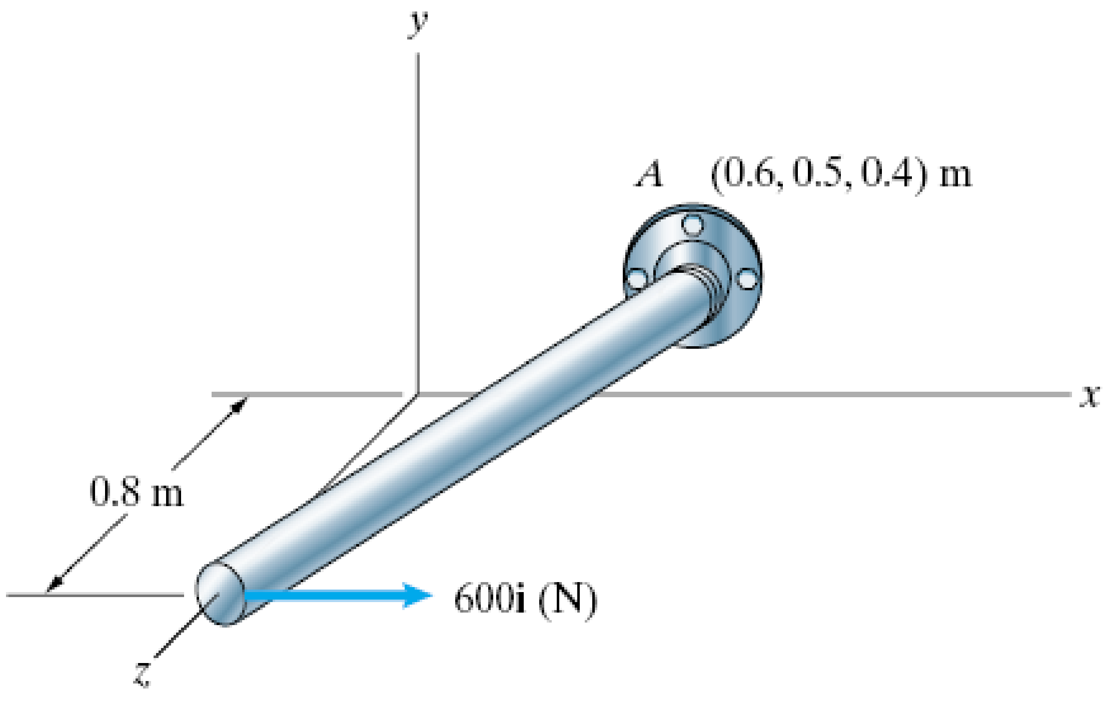
Similarly:

i X i = 0	**i X j = k**	**i X k = -j**
j X i = -k	**j X j = 0**	**j X k = i**
k X i = j	**k X j = -i**	**k X k = 0**

more ...

Example 1

Determine the magnitude of the moment of the 600 N force about A.



Solution 1

$\mathbf{r}_{\mathbf{AB}} = (\mathbf{-}0.6\mathbf{i} - 0.5\mathbf{j} + 0.4\mathbf{k})$ m

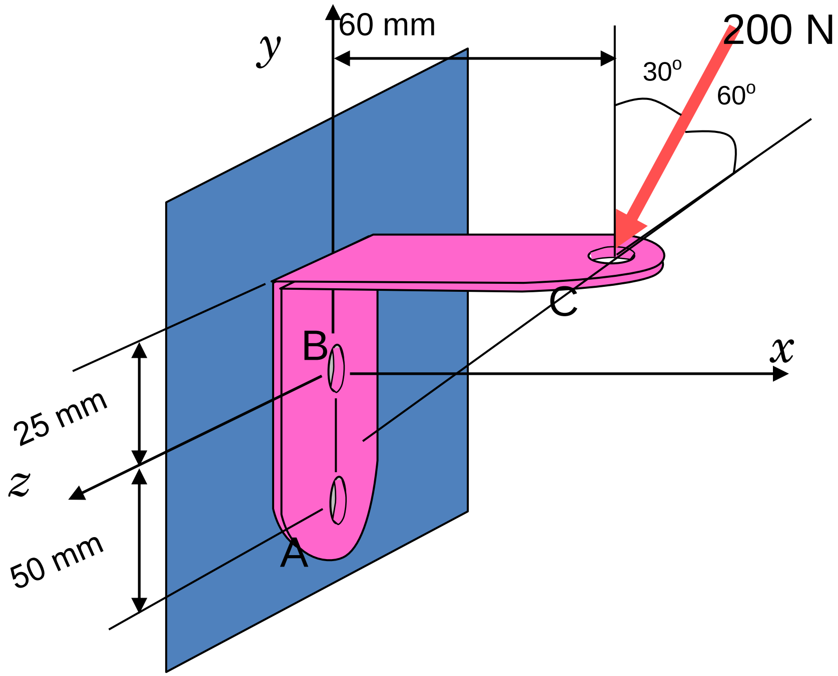
$\mathbf{M}_{\mathbf{A}} = \mathbf{r}_{\mathbf{AB}} \times \mathbf{F} = (\mathbf{-}0.6\mathbf{i} - 0.5\mathbf{j} + 0.4\mathbf{k}) \times (600\mathbf{i})$

$\mathbf{M}_{\mathbf{A}} = 240\ \mathbf{j} + 300\ \mathbf{k}$ N.m

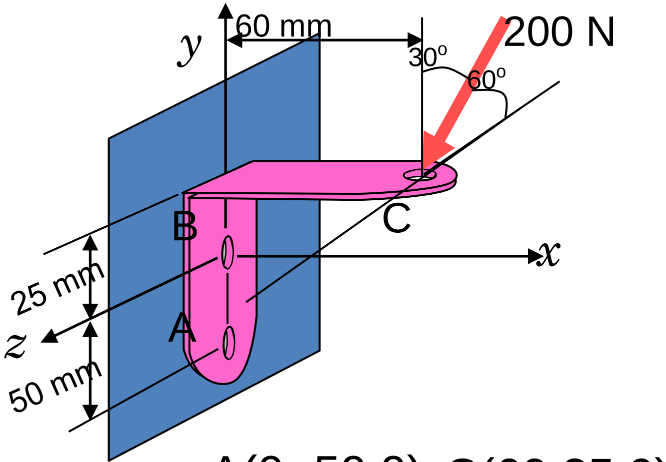
$M_A = 384$ N.m

Example 2

A 200-N force is applied as shown to the bracket ABC. Determine the moment of the force about A.



Solution 2



A(0,-50,0) C(60,25,0)

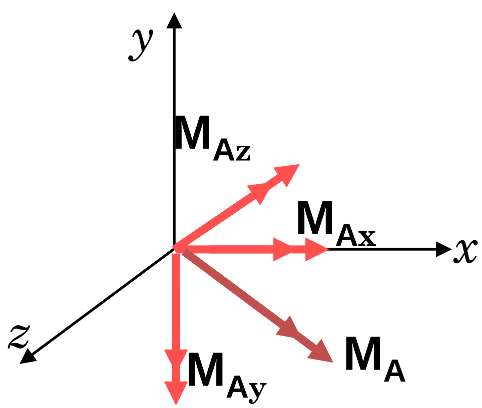
$\mathbf{r}_{C/A} = (0.06\ \mathbf{i} + 0.075\ \mathbf{j}\ )\ \text{m}$

$\mathbf{M}_A = \mathbf{r}_{C/A}\ \mathbf{x}\ \mathbf{F}$

$\mathbf{F} = -(200\ \text{N})\cos 30^o\ \mathbf{j} + (200\ \text{N})\cos 60^o\ \mathbf{k}$

$= -173.2\ \mathbf{j} + 100\ \mathbf{k}$

$\mathbf{M}_A = (0.06\ \mathbf{i} + 0.075\ \mathbf{j})\ \mathbf{x}\ (-173.2\ \mathbf{j} + 100\ \mathbf{k}) = -10.39\ \mathbf{k}\ - 6\ \mathbf{j}\ + 7.5\ \mathbf{i}$



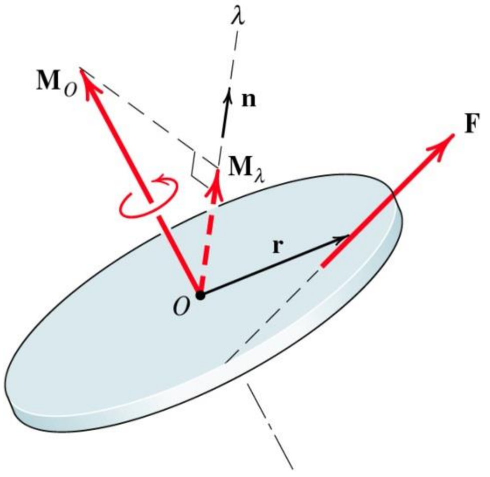
$\mathbf{M}_A = 7.5\ \mathbf{i} - 6\ \mathbf{j} - 10.39\ \mathbf{k}$

$M_A = 14.15\ \text{Nm}$

Moment of a force about an axis

First compute the moment of ***F*** about any arbitrary point (for example O) that lies on the λ axis using the cross product.

Now, find the component of M_λ along the axis λ using the dot product. Result will be scalar. So to find $\boldsymbol{M}_\lambda$ which is in vectoral form.



$$\vec{M}_\lambda = \underbrace{\left(\underbrace{\vec{r} \times \vec{F}}_{\vec{M}_o} \cdot \vec{n}_\lambda \right)}_{scalar} \vec{n}_\lambda$$

Moment of a force about an axis

Now, find the component of $\boldsymbol{M_a}$ along the axis a'-a using the dot product.

$$\boldsymbol{M}_\lambda = \boldsymbol{n}_\lambda \bullet \boldsymbol{M_O}$$

n $_\lambda$ represents the unit vector along the axis λ,

r is the position vector from any point on the λ axis to any point A on the line of action of the force, and

F is the force vector.

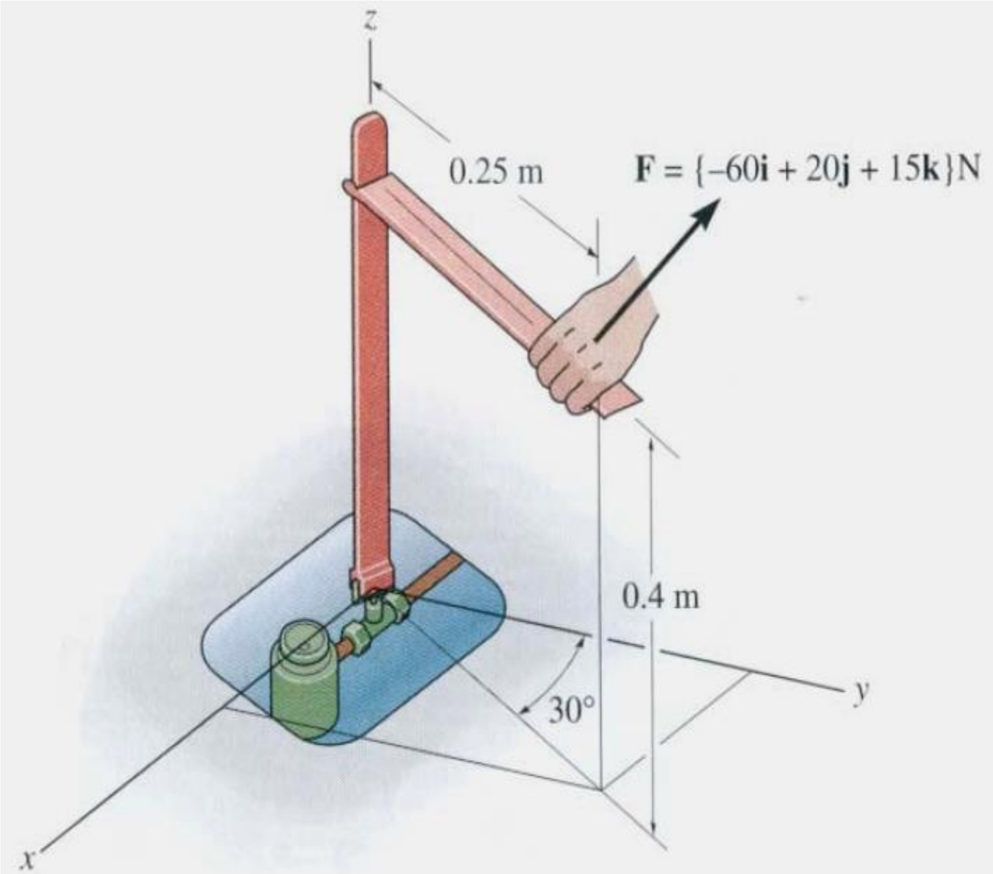
M $_\lambda$ **can also be obtained as**

$$\mathrm{M}_\lambda = \mathbf{n}_\lambda \cdot (\mathbf{r} \times \mathbf{F}) = \begin{vmatrix} n_{\lambda_x} & n_{\lambda_y} & n_{\lambda_z} \\ r_x & r_y & r_z \\ F_x & F_y & F_z \end{vmatrix}$$

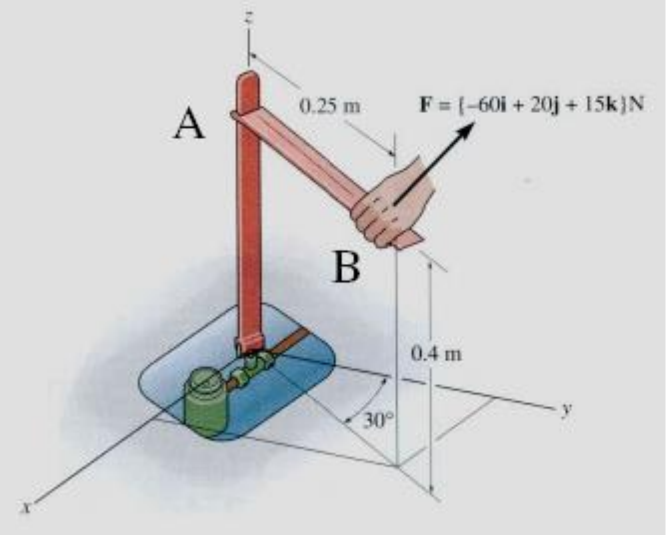
The above equation is also called the triple scalar product.

Example 3

A force is applied to the tool to open a gas valve. Find the magnitude of the moment of this force about the z axis of the valve.



Solution 3



$\boldsymbol{n} = 1\,\boldsymbol{k}$

$\boldsymbol{r}_{AB} = \{0.25 \sin 30° \,\boldsymbol{i} + 0.25 \cos 30° \,\boldsymbol{j}\}$ m
$= \{0.125\,\boldsymbol{i} + 0.2165\,\boldsymbol{j}\}$ m

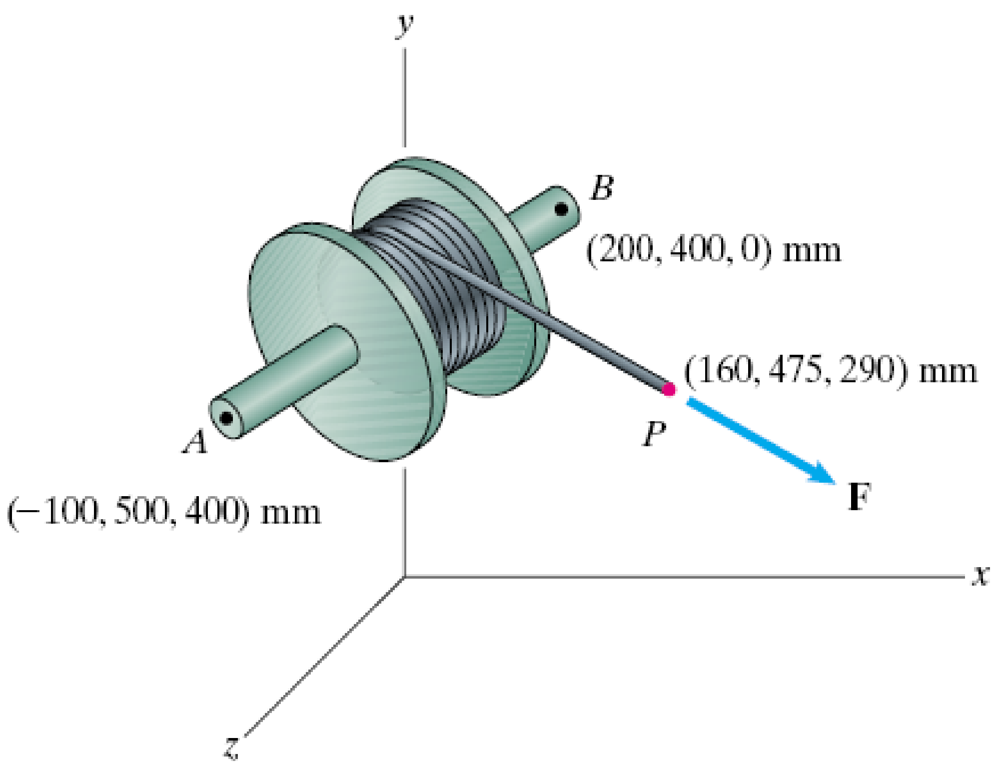
$\boldsymbol{F} = \{-60\,\boldsymbol{i} + 20\,\boldsymbol{j} + 15\,\boldsymbol{k}\}$ N

$M_z = \boldsymbol{n} \bullet (\boldsymbol{r}_{AB} \times \boldsymbol{F})$

$$M_z = \begin{vmatrix} 0 & 0 & 1 \\ 0.125 & 0.2165 & 0 \\ -60 & 20 & 15 \end{vmatrix}$$

$= 1\{0.125(20) - 0.2165(-60)\}$ N·m
$= 15.5$ N·m

Example 4



The magnitude of the force **F** is 0.2 N and its direction cosines are $\cos\theta_x = 0.727$, $\cos\theta y = -0.364$, and $\cos\theta z = 0.582$. Determine the magnitude of the moment of **F** about the axis *AB* of the spool.

Solution 4

$$\mathbf{r_{AB}} = (0.3\mathbf{i} - 0.1\mathbf{j} - 0.4\mathbf{k})\ \mathrm{m}$$

$$r_{AB} = \sqrt{(0.3)^2 + (0.1)^2 + (0.4)^2} = \sqrt{0.26}\, m$$

$$\mathbf{n_{AB}} = (0.3\mathbf{i} - 0.1\mathbf{j} - 0.4\mathbf{k}) / (0.26)^{1/2}\ \mathrm{m}$$

$$\mathbf{F} = 0.2\ (0.72\mathbf{i} - 0.364\mathbf{j} + 0.582\mathbf{k})\ \mathrm{m}$$

$$\mathbf{r_{AP}} = (0.26\mathbf{i} - 0.025\mathbf{j} - 0.11\mathbf{k})\ \mathrm{m}$$

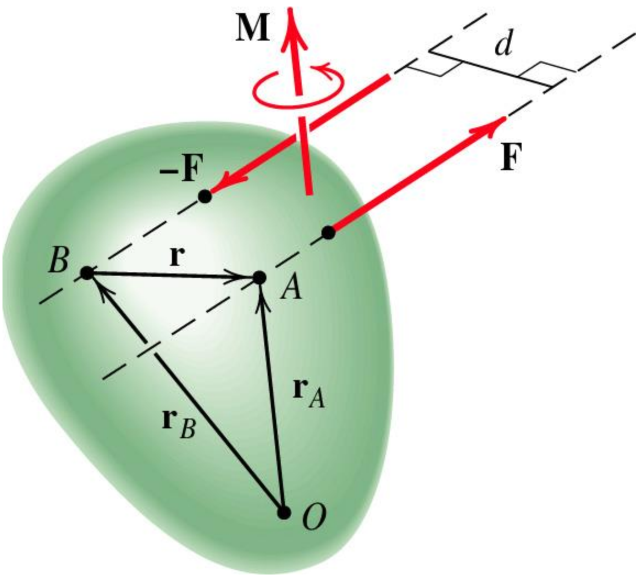
$$\vec{M}_A = \begin{vmatrix} i & j & k \\ 0.26 & -0.025 & -0.11 \\ 0.727 & -0.364 & 0.582 \end{vmatrix}$$

Solution 4

$$M_{AB} = \mathbf{M_A} \cdot \mathbf{n_{AB}}$$

$$\boxed{M_{AB} = \frac{0.2\ \text{N}}{\sqrt{0.26}} \begin{vmatrix} 0.3 & -0.1 & -0.4 \\ 0.26\ \text{m} & -0.025\ \text{m} & -0.11\ \text{m} \\ 0.727 & -0.364 & 0.582 \end{vmatrix} = 0.0146\ \text{N-m}}$$

Couple in 3D



The vector **r** runs from any point *B* on the line of action of –**F** to any point *A* on the line of action of **F**. Points *A* and *B* are located by positions vectors $\mathbf{r}_A$ and $\mathbf{r}_B$ from any point O. The combined moment of two forces about O is

$$\vec{M} = \vec{r}_A \times \vec{F} + \vec{r}_B \times \left(-\vec{F}\right) = \left(\vec{r}_A - \vec{r}_B\right) \times \vec{F}$$

Couple in 3D

$$\vec{M} = \vec{r}_A \times \vec{F} + \vec{r}_B \times \left(-\vec{F}\right) = \left(\vec{r}_A - \vec{r}_B\right) \times \vec{F}$$

However, $\mathbf{r}_A - \mathbf{r}_B = r$, so moment of the couple becomes

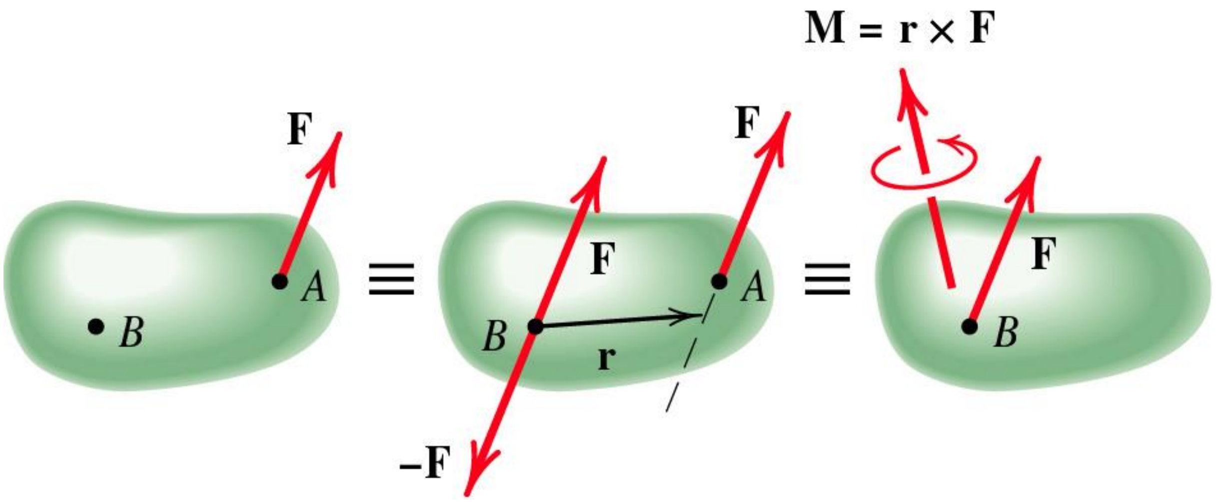
$$\vec{M} = \vec{r} \times \vec{F}$$

Thus, moment of a couple is same about all poins. The magnitude of **M** is M=Fd, where d is the perpendicular distance between the lines of action of the two forces.

The Moment of a couple is **a free vector**, whereas the moment of a force about a point is **a sliding vector** whose direction is along the axis through the point.

A couple produce **a pure rotation** of the body about an axis normal to the plane of the forces which constitude the couple.

Equivalent force-couple system

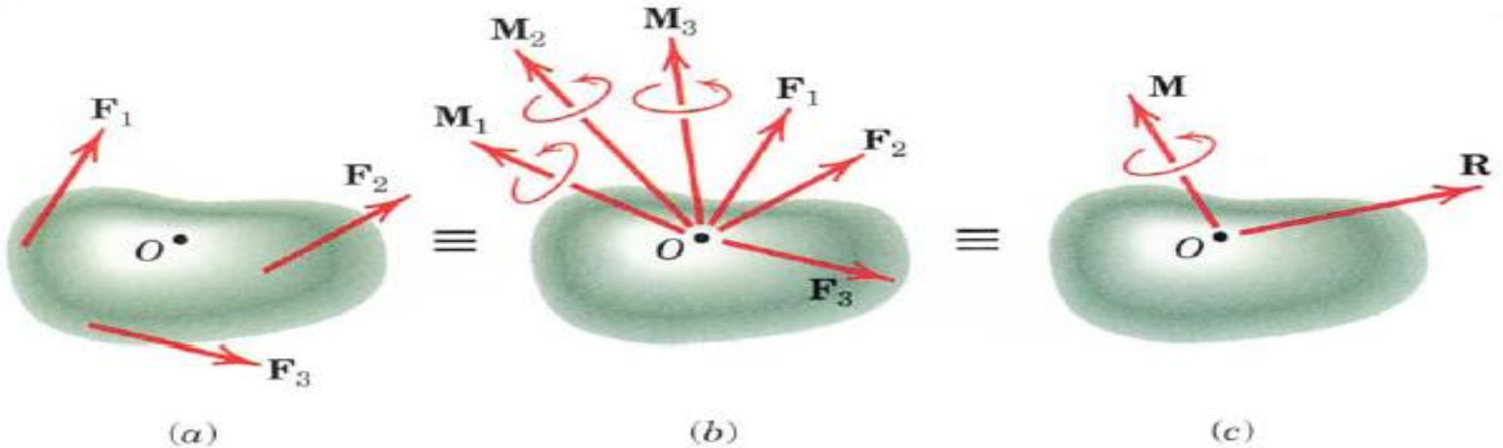


Like 2-D systems, a force can be replaced by its equivalent force-couple system in three dimensions.

As seen in the figure, the force **F** acting on a rigid body at point *A* is replaced by an equal force at point *B* and the couple **M**= **r**x**F**.

r is a vector which runs from *B* to any point on the line of action of the original force passing through *A*.

Resultant



$$\mathbf{R} = \mathbf{F}_1 + \mathbf{F}_2 + \mathbf{F}_3 + \cdots = \Sigma\mathbf{F}$$

$$\mathbf{M} = \mathbf{M}_1 + \mathbf{M}_2 + \mathbf{M}_3 + \cdots = \Sigma(\mathbf{r} \times \mathbf{F})$$

$$R_x = \Sigma F_x \qquad R_y = \Sigma F_y \qquad R_z = \Sigma F_z$$

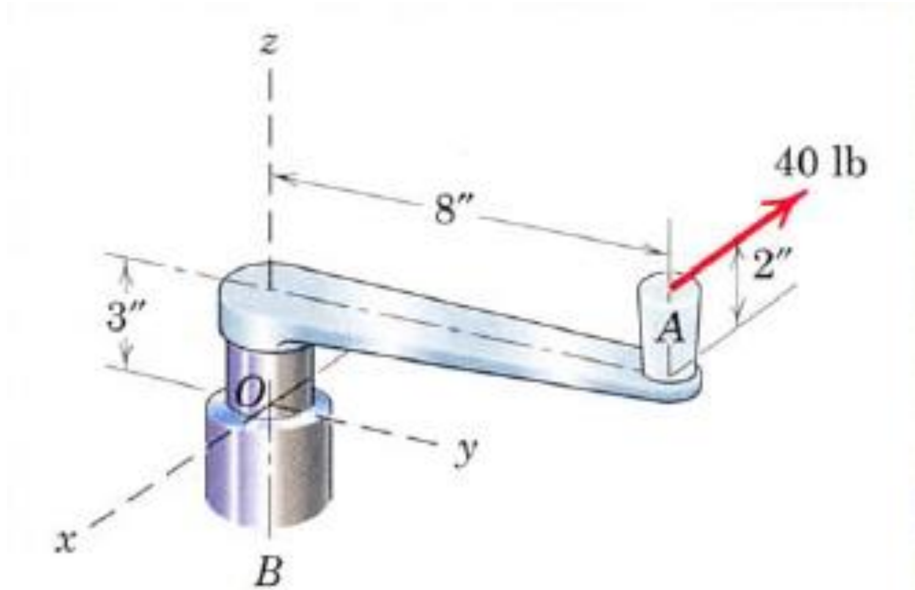
$$R = \sqrt{(\Sigma F_x)^2 + (\Sigma F_y)^2 + (\Sigma F_z)^2}$$

$$\mathbf{M}_x = \Sigma(\mathbf{r} \times \mathbf{F})_x \qquad \mathbf{M}_y = \Sigma(\mathbf{r} \times \mathbf{F})_y \qquad \mathbf{M}_z = \Sigma(\mathbf{r} \times \mathbf{F})_z$$

$$M = \sqrt{M_x{}^2 + M_y{}^2 + M_z{}^2}$$

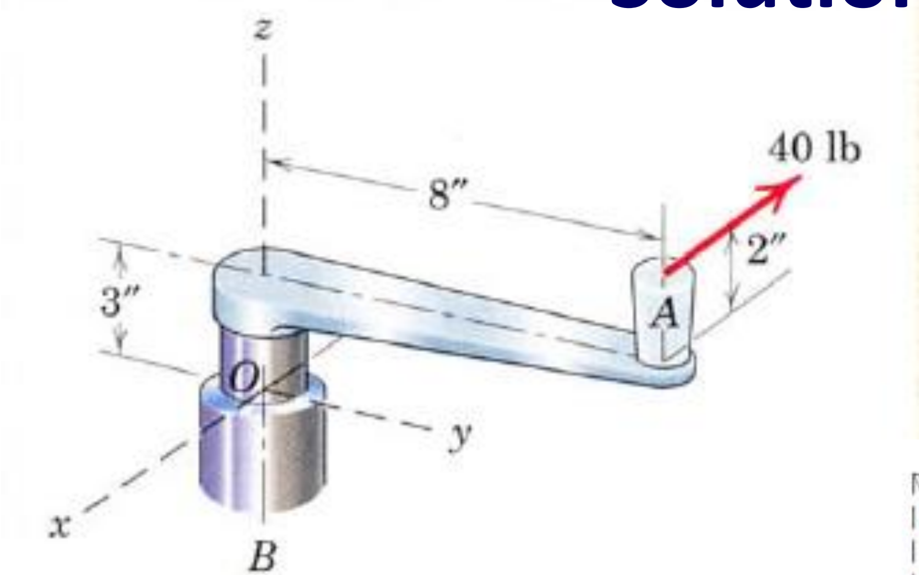
Example 5

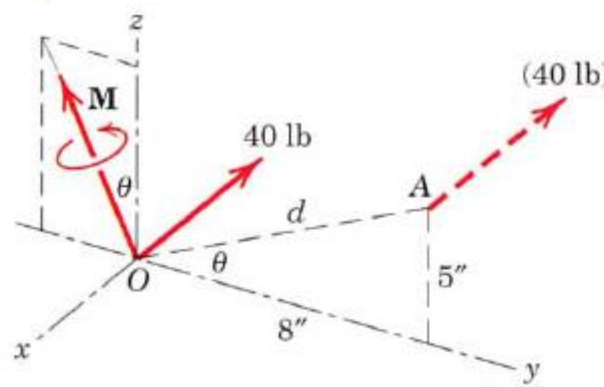
A force of 40 lb is applied at A to the handle of the control lever which is attached to the fixed shaft *OB*. In determining the effect of the force on the shaft at a cross section such as that at O, we may replace the force by an equivalent force at O and a couple. Describe this couple as a vector M.



Answer: **M**)= -200 **j** + -320 **k lb.in**

Solution 5





M = $\mathbf{r}_{OA}$ x F

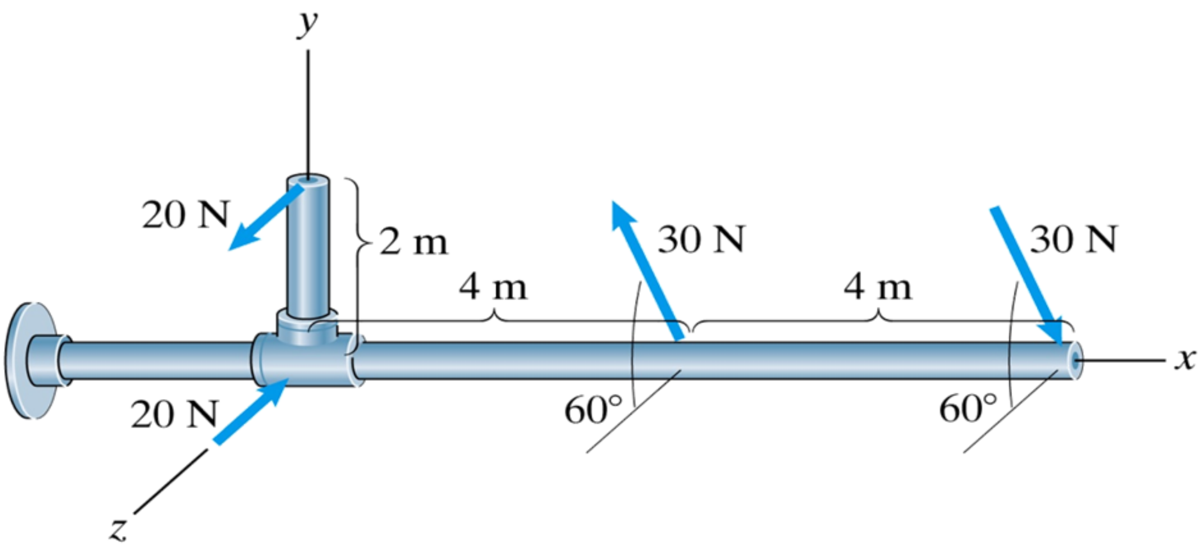
$\mathbf{r}_{\mathbf{OA}} = 8\mathbf{j} + 5\ \mathbf{k}$ in

$\mathbf{F}$ = -40 $\mathbf{i}$ lb

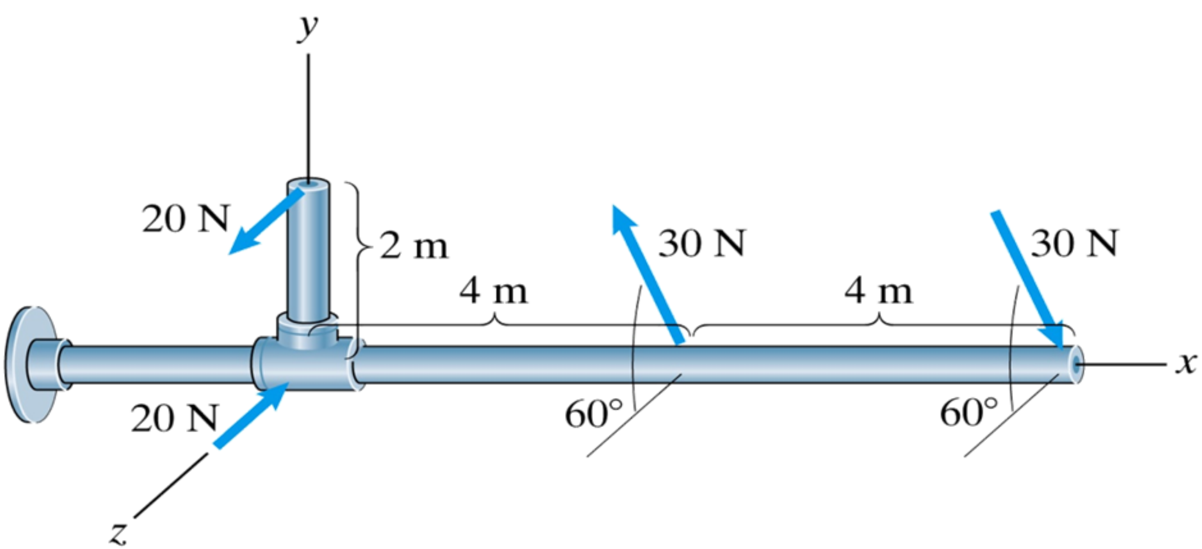
M =(8**j** + 5 **k**)**x (**-40 **i**)= -200 **j** + -320 **k lb.in**

Example 6

Determine the sum of the moments exerted on the pipe in figure by the 2 couples.



Solution 6



Consider the 20-N couple.

The magnitude of the moment of the couple is:

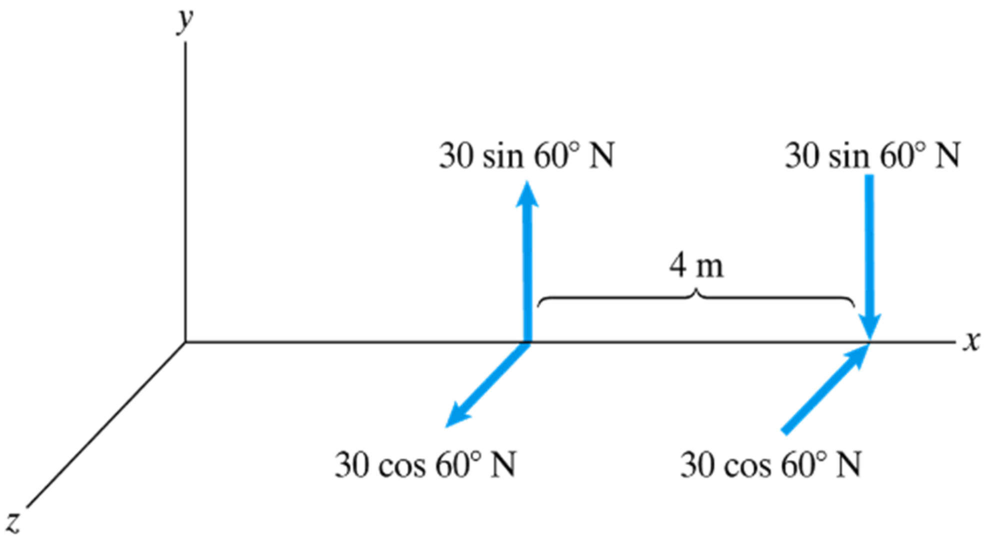
$$(2\text{ m})(20\text{ N}) = 40\text{ N-m}$$

The direction of the moment vector is perpendicular to the y-z plane & the right-hand rule indicates that it points in the positive x axis direction.

The moment of the 20-N couple is $40\mathbf{i}$ (N-m).

Solution 6

By resolving the 30-N forces into y & z components, we obtain the 2 couples:

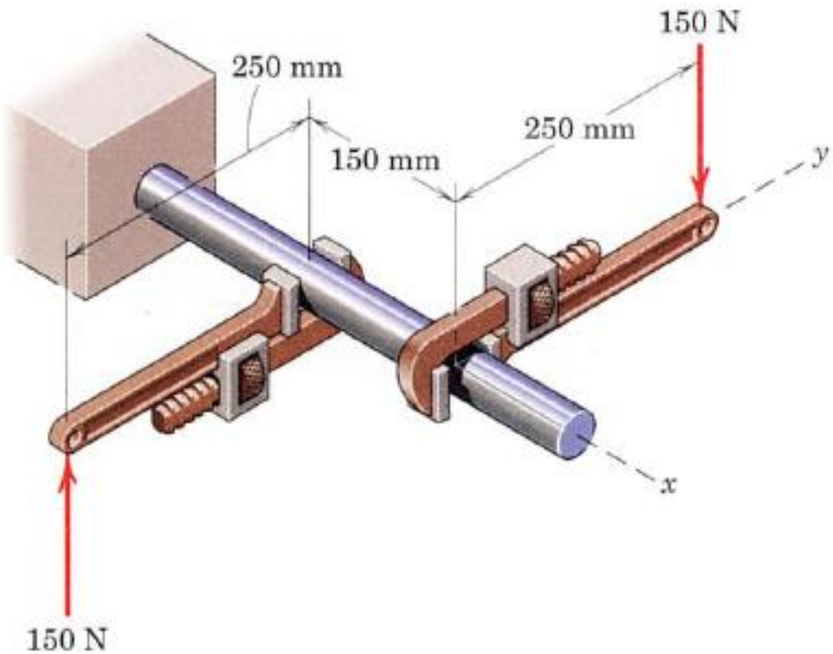


the couple formed by the y components is $-(30 \sin 60°)(4)\mathbf{k}$ (N-m)
the couple formed by the z components is $(30 \cos 60°)(4)\mathbf{j}$ (N-m).

The sum is therefore:

$$\Sigma M = 40\mathbf{i} + (30 \cos 60°)(4)\mathbf{j} - (30 \sin 60°)(4)\mathbf{k} \text{ (N-m)}$$
$$= 40\mathbf{i} + 60\mathbf{j} - 104\mathbf{k} \text{ (N-m)}$$

Example 7



The two forces acting on the handles of the pipe wrenches constitute a coupe **M**. Express the couple vector.

Answer: **M** = -75 **i** + 22.5 **j** N.m