

ME 209

Numerical Methods

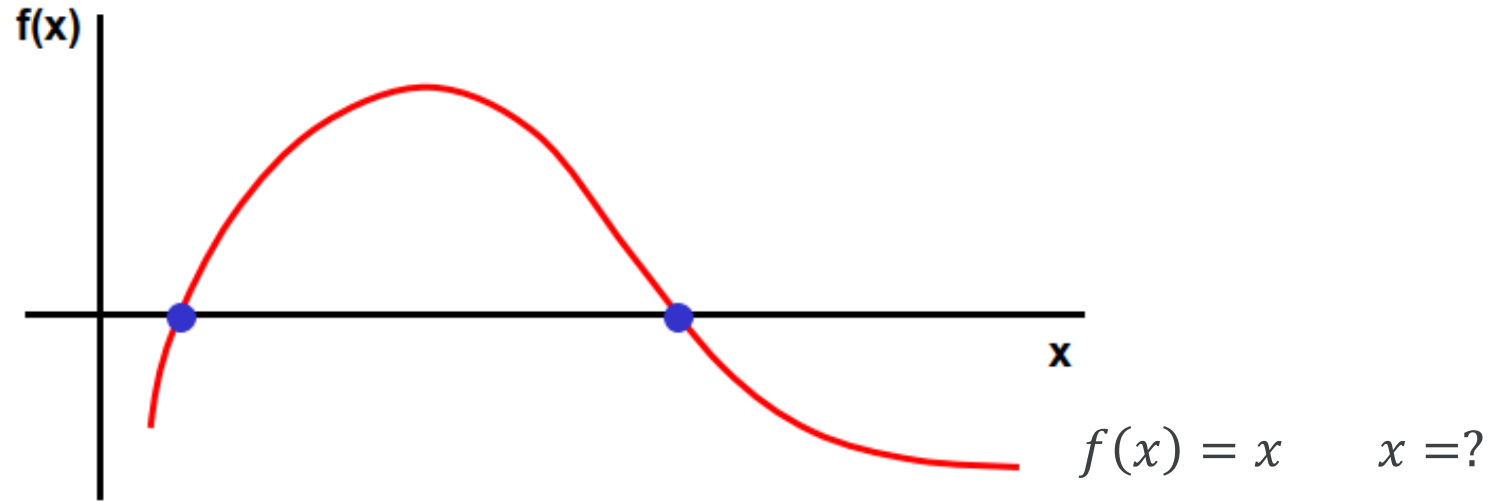
3. Finding Roots of the Equations: Bracketing Methods

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3.1. INTRODUCTION



- Finding the root of an equation $f(x)$ or solving that equation means determining the x values that make the equation zero. For this reason, the roots of equations are sometimes called the zeros of these equations.
- We know how to find the roots of 2nd degree polynomial function $f(x) = ax^2 + bx + c = 0$

by using the equation
$$x = \frac{-b \mp \sqrt{b^2 - 4ac}}{2a}$$

3.2. FINDING ROOTS OF EQUATIONS

- However, there are many complex equations whose roots we cannot easily find analytically. In such cases, numerical methods can be used as powerful alternatives in finding the roots of equations.
- Numerical methods commonly used in finding the roots of equations can be examined in two groups. These are: ***Bracketing methods*** and ***open methods***.

Bracketing (closed) methods:

- Bisection Method
- False-Position Method

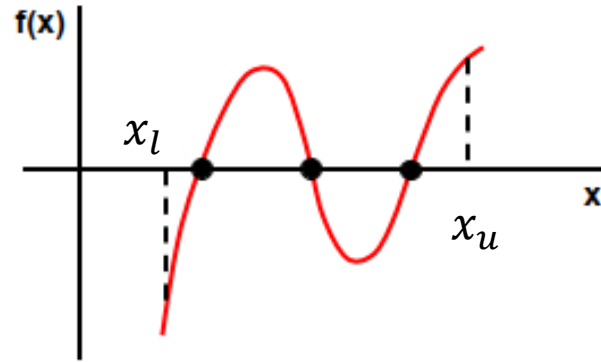
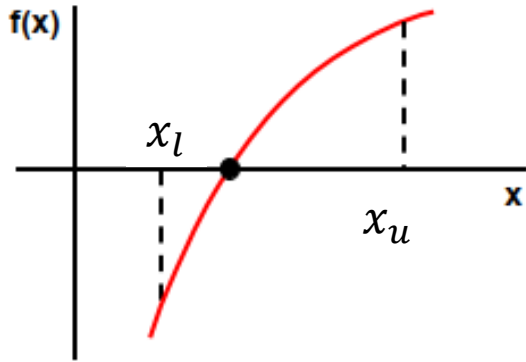
Open methods:

- One-point Iteration Method
- Newton-Raphson Method
- Secant Method

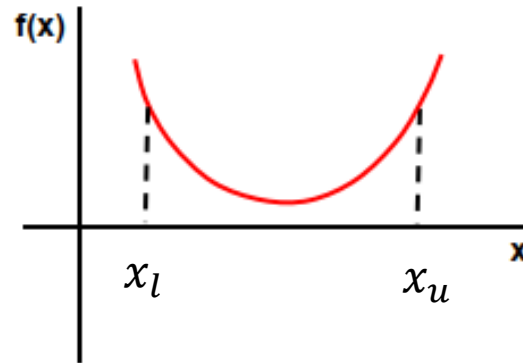
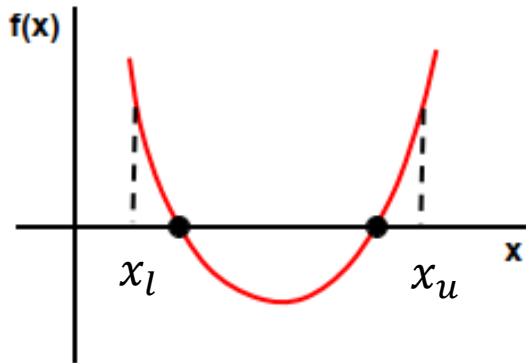
3.3. Bracketing Methods

- These techniques deal with techniques that take use of the fact that a function frequently changes sign close to a root.
- And these techniques are called bracketing methods because ***two initial guesses for the root are required.***
- As the name implies, these guesses must “bracket,” or be on either side of, the root.
- The particular methods described herein employ different strategies to systematically reduce the width of the bracket and, hence, focus on the correct answer.

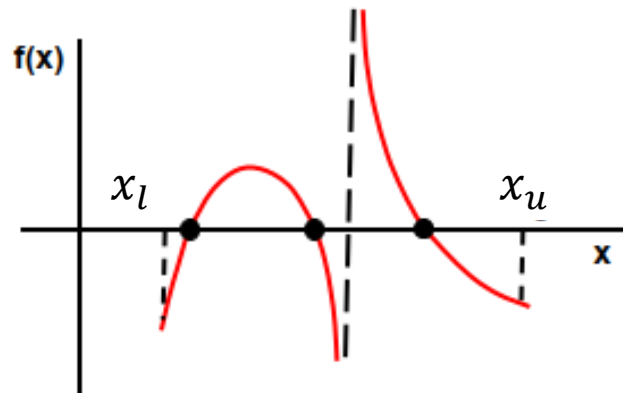
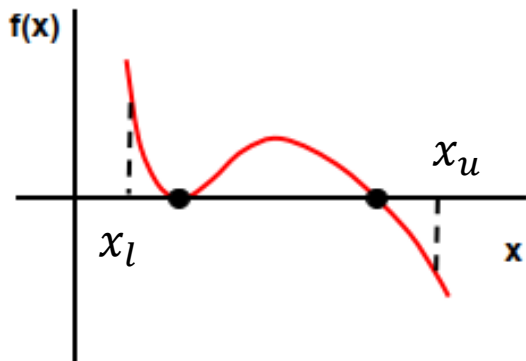
3.3.1. General Idea of Bracketing Methods



RULE 1: If $f(x)$ is continuous on (x_l, x_u) and $f(x_l) * f(x_u) < 0$ then at least one root exists in (x_l, x_u) .



RULE 2: IF $f(x_l) * f(x_u) > 0$ THEN
there are: (i) even number of roots,
(ii) no roots.



VIOLATIONS:

- (i) multiple roots,
- (ii) discontinuities.

Question: Does a sign change guarantee a root?

- $f(x) = (x - 1)^2 \rightarrow$ function is positive at $[0,2]$. Is the root included in this range?

YES. The function is also continuous in $[0,2]$ range.

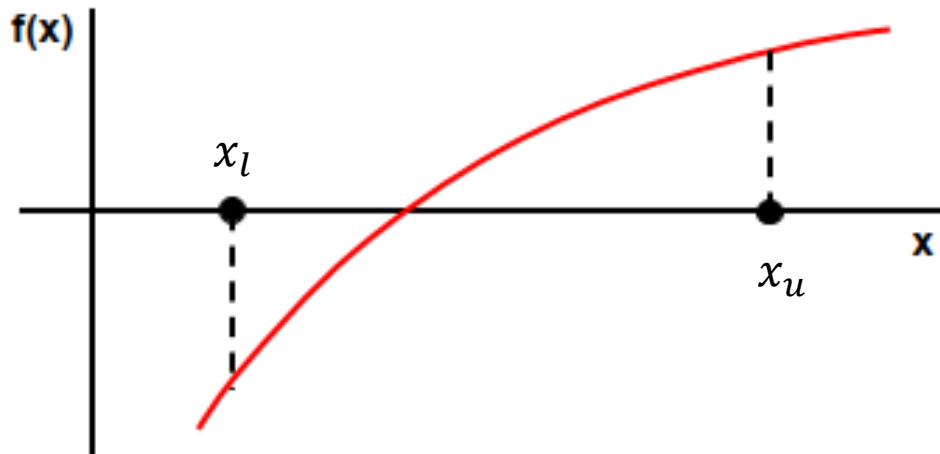
- $f(x) = \frac{1}{x-1} \rightarrow$ function is positive at $[0,2]$. Is the root included in this range?

NO. The function is NOT continuous in $[0,2]$ range. At $x = 1$ the function is infinite (discontinuity).

We should be aware of the discontinuity of the function!

3.3.2. THE BISECTION METHOD

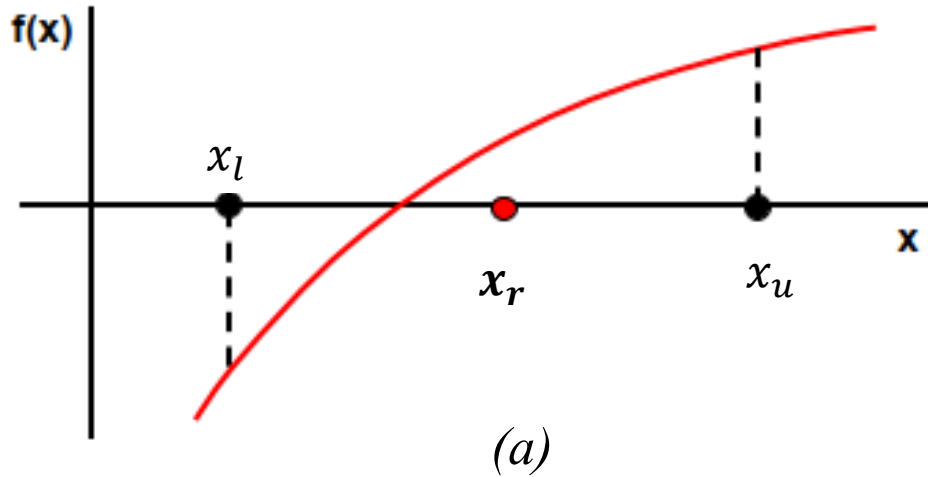
The **bisection method** is one type of incremental search method in which the interval is always divided in half. If a function changes sign over an interval, the function value at the midpoint is evaluated.



- **Step 1:** Choose two initial estimations, x_{LOWER} (x_l) and x_{UPPER} (x_u).
- They should bracket the root, i.e.

$$f(x_l) * f(x_u) < 0$$

- Check whether the function $f(x)$ is continuous in this interval.

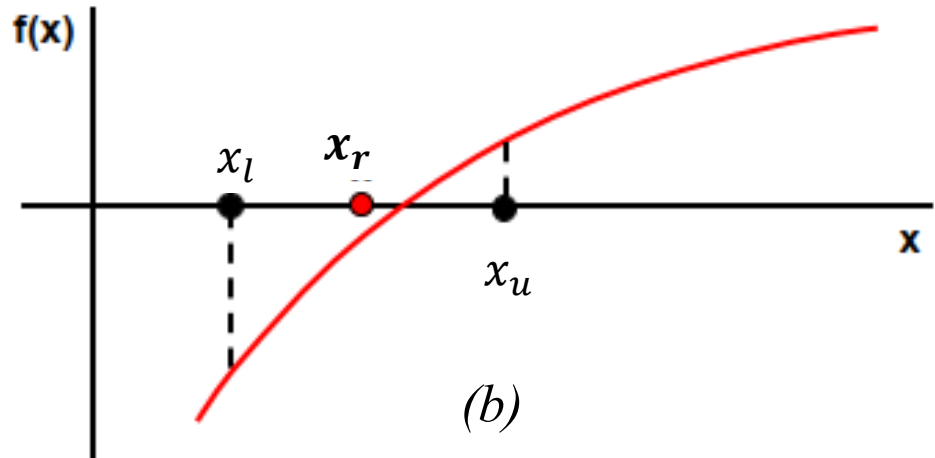


Step 2: Estimate the root as a midpoint of this interval

$$x_r = \frac{(x_l + x_u)}{2}$$

Step 3: Determine the interval which contains the root,

(a) IF $f(x_l) * f(x_r) < 0$ THEN the root is between x_L and x_r ,
Therefore, set $x_u = x_r$ and RETURN to step 2.



(b) IF $f(x_l) * f(x_r) > 0$ THEN the root is between x_r and x_U .
Therefore, set $x_l = x_r$ and RETURN to step 2.

(c) IF $f(x_l) * f(x_r) = 0$, the root is equals x_r ; TERMINATE the computation

Termination Criteria and Error Estimates

We need a termination criteria to end the iteration. Here, an approximate percent relative error ε_a can be calculated as:

$$\varepsilon_a = \left| \frac{x_r^{new} - x_r^{old}}{x_r^{new}} \right| 100\%$$

When the approximate percent relative error $|\varepsilon_a|$ falls below the specified percent relative error ε_s or tolerance, we could terminate the bisection method.

Example 3.1: Find the square root of 11. (Tolerance value: $|\varepsilon_s| = 0.5\%$)

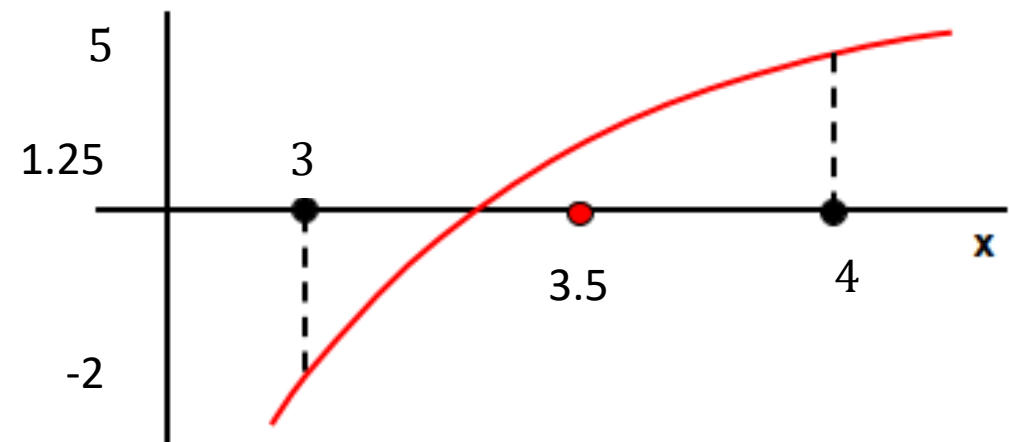
$$x^2 = 11 \rightarrow f(x) = x^2 - 11 \quad (\text{exact solution is } 3.31662479)$$

Choose initial estimates: $3^2 = 9 < 11$ and $4^2 = 16 > 11 \rightarrow x_l = 3$ and $x_u = 4$

1st iteration: $x_r = \frac{x_l + x_u}{2} = \frac{3+4}{2} \rightarrow x_r = 3.5$ and $f(3.5) = 3.5^2 - 11 = 1.25$

$$f(x_l) * f(x_r) = -2 * 1.25 \rightarrow -2.5 < 0 \quad (\text{root is between } x_l \text{ and } x_r)$$

SET $x_u = x_r = 3.5$ and CONTINUE to iterate.



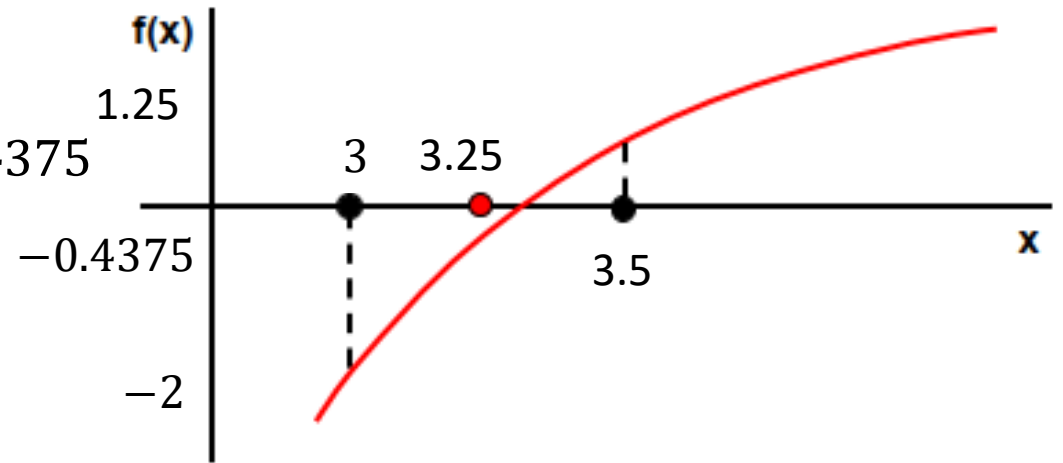
2nd iteration: $x_l = 3, x_u = 3.5$

$$x_r = \frac{3+3.5}{2} \rightarrow x_r = 3.25 \text{ and } f(3.25) = 3.25^2 - 11 = -0.4375$$

$$f(x_l) * f(x_r) = -2 * -0.4375 = 0.875 > 0$$

(root is between x_r and x_u)

SET $x_l = x_r = 3.25$ and CONTINUE to iterate.



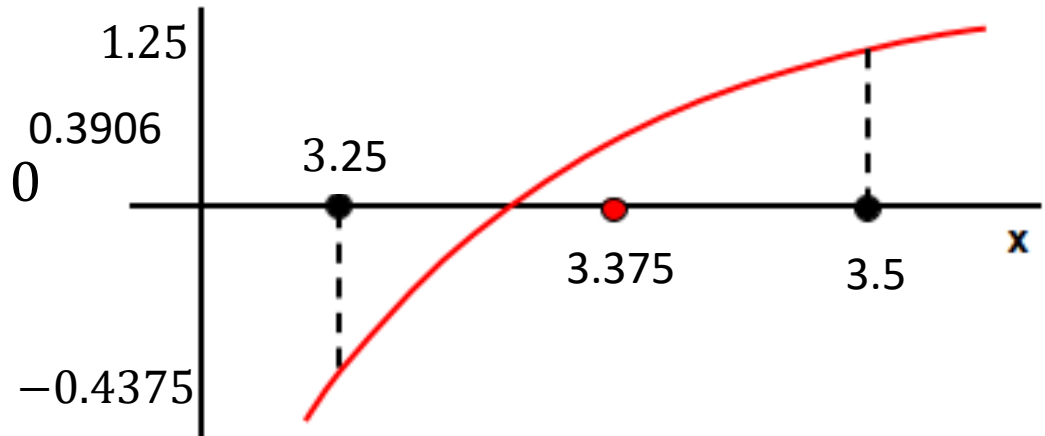
3rd iteration: $x_l = 3.25, x_u = 3.5$

$$x_r = \frac{3.25+3.5}{2} \rightarrow x_r = 3.375 \text{ and } f(3.375) = 0.390625$$

$$f(x_l) * f(x_r) = -0.4375 * 0.390625 = -0.17089 < 0$$

(root is between x_l and x_r)

SET $x_u = x_r = 3.375$ and CONTINUE to iterate.



4th iteration: $x_l = 3.25, x_u = 3.375$

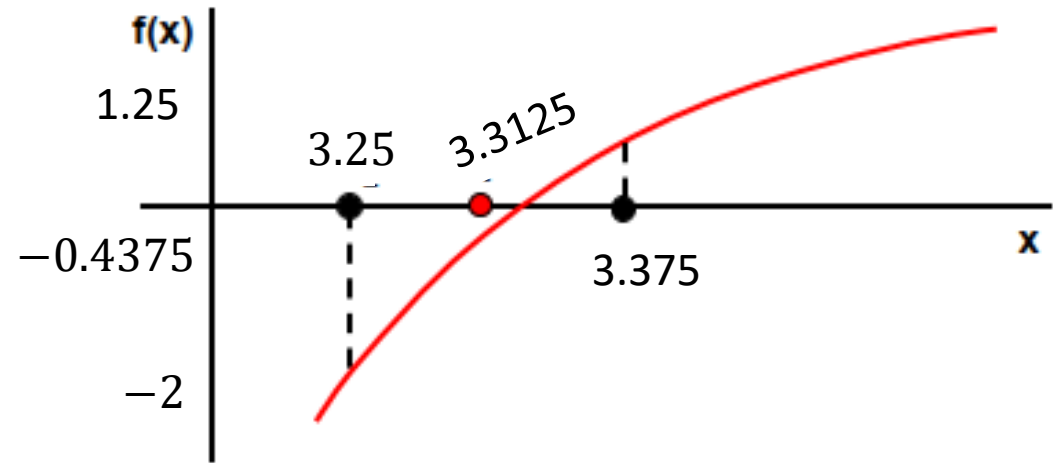
$$x_r = \frac{3.25+3.375}{2} \rightarrow x_r = 3.3125$$

$$\text{and } f(3.3125) = 3.3125^2 - 11 = -0.027343$$

$$-0.4375 * -0.027343 = 0.011963 > 0$$

(root is between x_r and x_u)

SET $x_l = x_r = 3.3125$ and CONTINUE to iterate.



5th iteration: $x_l = 3.3125, x_u = 3.375$

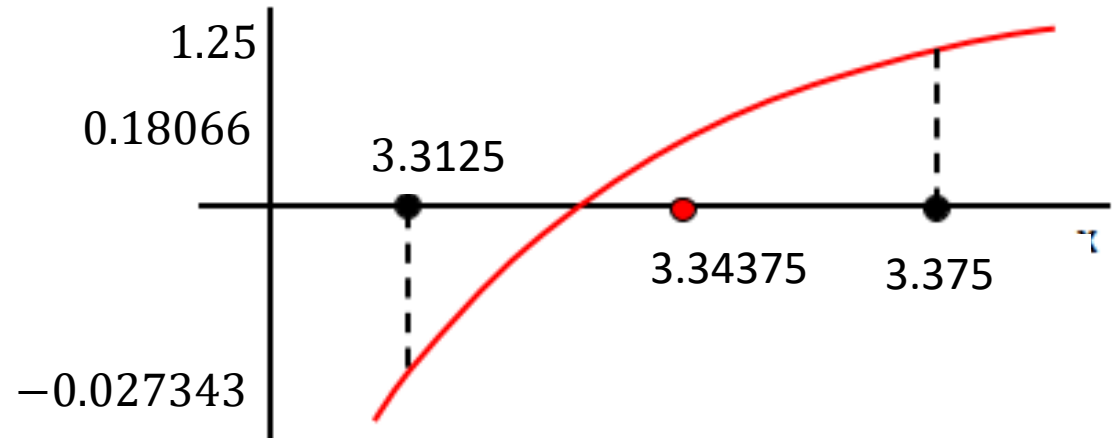
$$x_r = \frac{3.3125+3.375}{2} \rightarrow x_r = 3.34375 \text{ and}$$

$$f(3.34375) = 0.18066$$

$$-0.027343 * 0.18066 = -0.01068 < 0$$

(root is between x_l and x_r)

SET $x_u = x_r = 3.34375$ and CONTINUE to iterate.

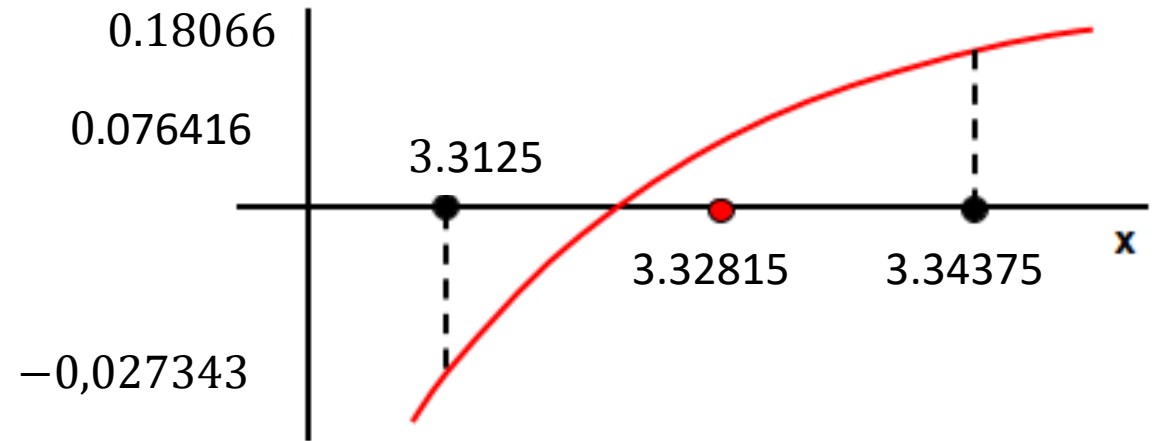


6th iteration: $x_l = 3.3125, x_u = 3.34375$

$$x_r = \frac{3.3125 + 3.34375}{2} \rightarrow x_r = 3.32815 \text{ and } f(3.32815) = 0.076416$$

$$-0.027343 * 0.076416 = -0.002089 < 0$$

(root is between x_l and x_r)



Iteration	x_l	x_r	x_u	$f(x_l)$	$f(x_r)$	$f(x_l)f(x_r)$	$ \varepsilon_a $ (%)
1	3	3.5	4	-2	1.25	-2.5	-
2	3	3.25	3.5	-2	-0.4375	0.875	7.69
3	3.25	3.375	3.5	-0.4375	0.390625	-0.17089	3.70
4	3.25	3.3125	3.375	-0.4375	-0.027343	0.011963	1.88
5	3.3125	3.34375	3.375	-0.027343	0.18066	-0.004942	0.93
6	3.3125	3.32815	3.34375	-0.027343	0.076582	-0.0020939	0.468

$$\sqrt{11} \approx 3.32815$$

$$|\varepsilon_a| < |\varepsilon_s|$$

Example 3.2 Calculate root of following polynomial

$$f(x) = x^3 - x^2 - 10x - 8 = 0$$

Finding the roots within the interval $3.75 \leq x \leq 5.00$ to a relative accuracy as an absolute value between successive iterations of 0.01.

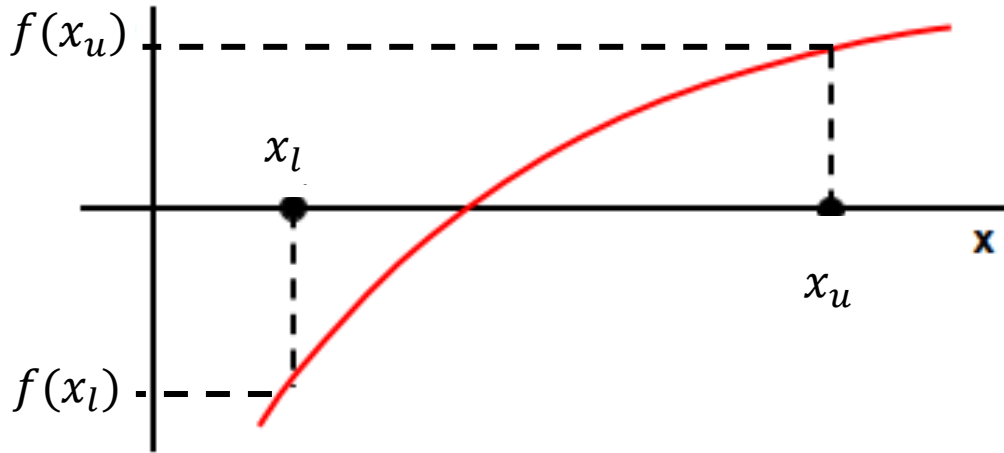
Check: $f(x=3.75) = -6.828125$

$f(x=5.00) = 42$

$f_l \times f_u = -286.44 < 0$

Iteration i	x_l	x_r	x_u	$f(x_l)$	$f(x_m)$	$f(x_u)$	Subinterval Containing Root		Error ϵ
							$f(x_l)f(x_r)$	$f(x_r)f(x_u)$	
1	3.750	4.375	5.000	-6.830	12.850	42.000	-	+	-
2	3.750	4.062	4.375	-6.830	1.903	12.850	-	+	0.313
3	3.750	3.906	4.062	-6.830	-2.724	1.903	+	-	0.156
4	3.906	3.984	4.062	-2.724	-0.477	1.903	+	-	0.078
5	3.984	4.023	4.062	-0.477	0.696	1.903	-	+	0.039
6	3.984	4.004	4.023	-0.477	0.120	0.696	-	+	0.019
7	3.984	3.994	4.004	-0.477	-0.180	0.120	+	-	0.010

3.3.3. FALSE-POSITION METHOD



- **Step 1:** Choose two initial estimations, x_{LOWER} (x_l) and x_{UPPER} (x_u).
- They should bracket the root, i.e.

$$f(x_l) * f(x_u) < 0$$

- **Step 2:** Using similar triangles, the intersection of the straight line with the x axis can be estimated as

$$x_r = \frac{x_u f(x_l) - x_l f(x_u)}{f(x_l) - f(x_u)}$$

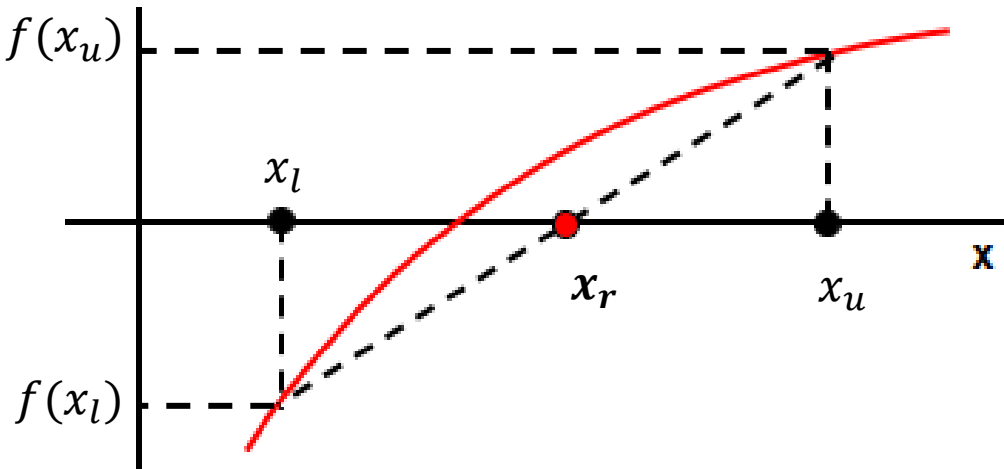
or

$$x_r = x_u - \frac{f(x_u)(x_l - x_u)}{f(x_l) - f(x_u)}$$

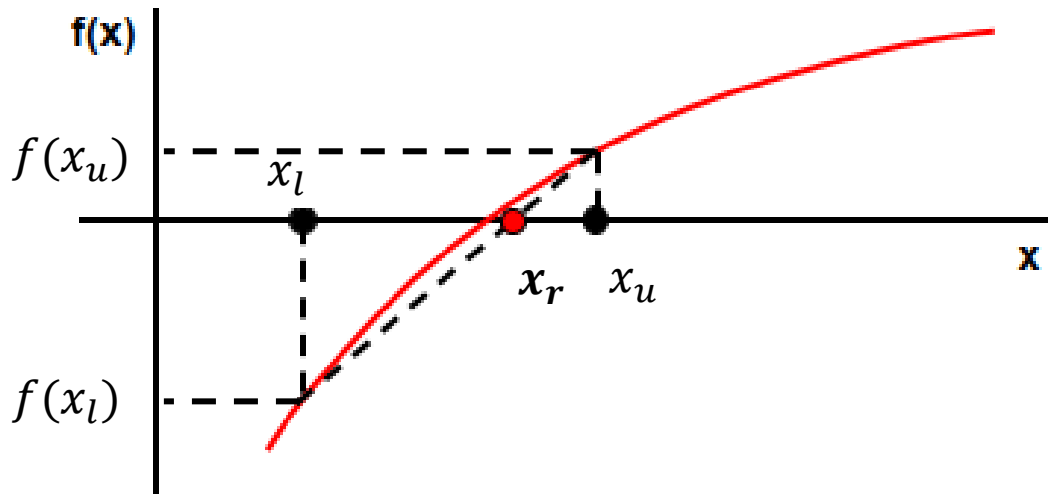
- Determine the interval which contains the root:

IF $f(x_l) * f(x_r) < 0$ root is between x_l and x_r

ELSE root is between x_r and x_u

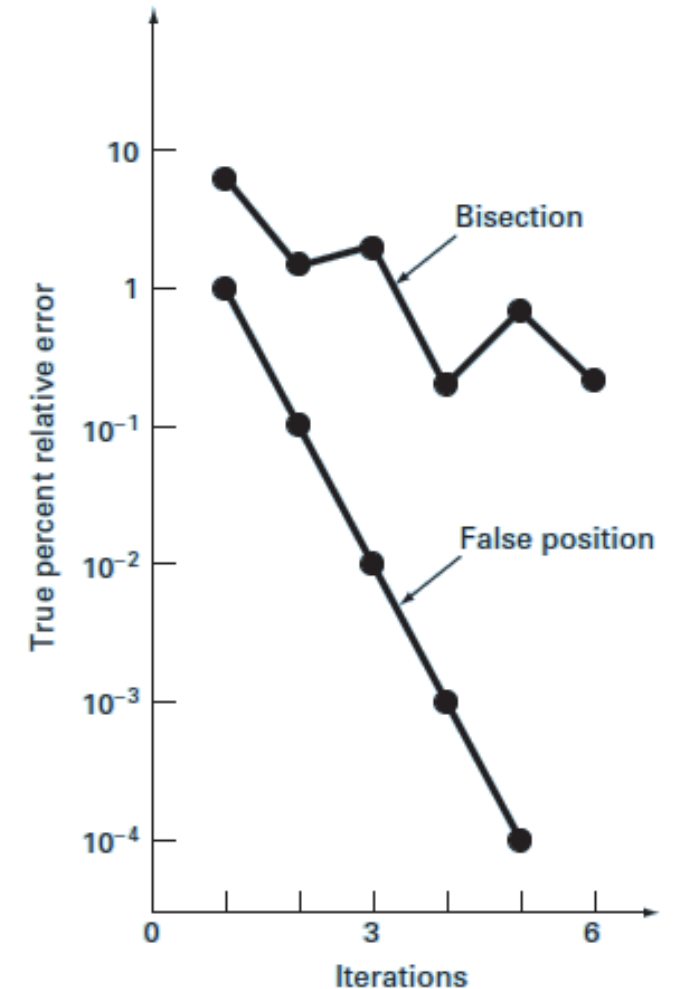


3.3.3. FALSE-POSITION METHOD



- **Step 3:** Estimate a new root in this interval,
Stop when the specified tolerance is reached.

- False-position method always converges to the true root,
- $f(x_l) * f(x_u) < 0$ is true if the interval has odd number of roots, not necessarily one root.
- The *false-position method* generally converges faster than the *bisection method*.



Example 3.2: Find the square root of 11 by using false-position method. (Tolerance value: $|\varepsilon_s| = 0.5\%$)

$$x^2 = 11 \rightarrow f(x) = x^2 - 11 \quad (\text{exact solution is } 3.31662479)$$

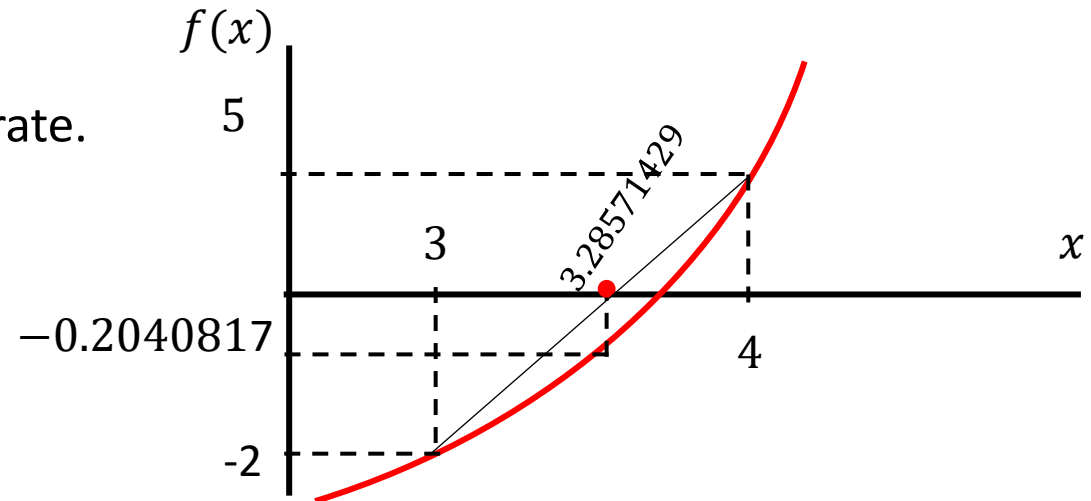
Choose initial estimates: $3^2 = 9 < 11$ and $4^2 = 16 > 11 \rightarrow x_l = 3$ and $x_u = 4$

1st iteration: $x_r = x_u - \frac{f(x_u)(x_l - x_u)}{f(x_l) - f(x_u)} \rightarrow x_r = 3.28571429$ and

$$f(3.28571429) = 3.28571429^2 - 11 = -0.2040817$$

$f(x_l) * f(x_r) = -2 * -0.2040817 \rightarrow 0.408163 > 0$ (root is between x_r and x_u)

SET $x_l = x_r = 3.28571429$ and CONTINUE to iterate.



2nd iteration: $x_l = 3.28571429$, $x_u = 4$

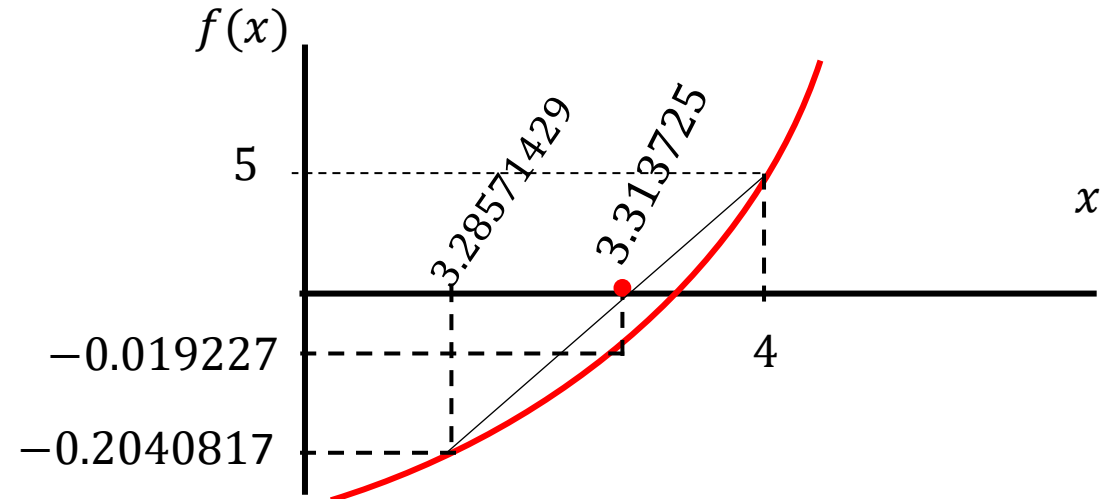
$$x_r = x_u - \frac{f(x_u)(x_l - x_u)}{f(x_l) - f(x_u)} \rightarrow x_r = 3.313725 \text{ and}$$

$$f(3.313725) = 3.313725^2 - 11 = -0.019227$$

$$f(x_l) * f(x_r) = -0.2040817 * -0.019227$$

$$\rightarrow 0.00392 > 0 \text{ (root is between } x_r \text{ and } x_u)$$

SET $x_l = x_r = 3.313725$ and CONTINUE to iterate.



3rd iteration: $x_l = 3.313725$, $x_u = 4$

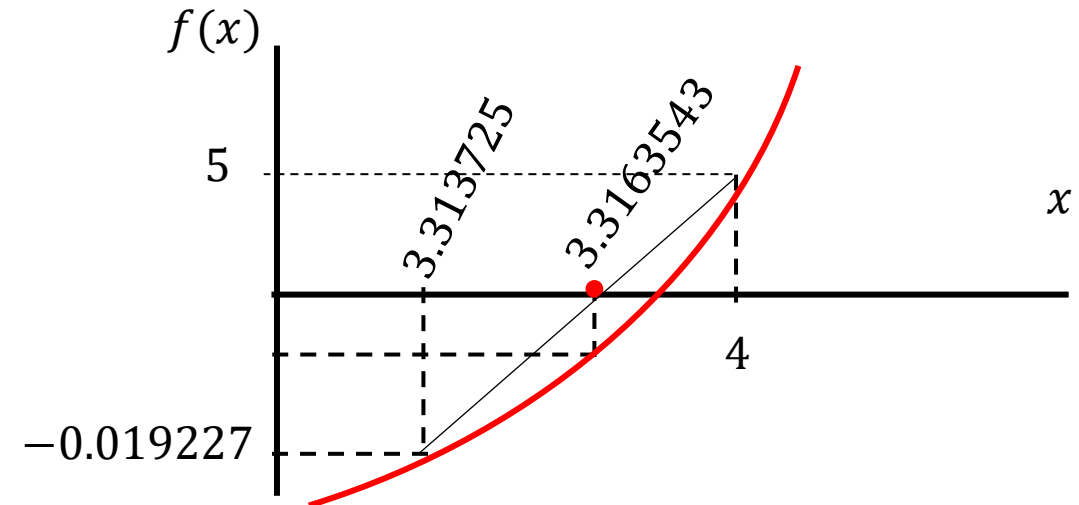
$$x_r = x_u - \frac{f(x_u)(x_l - x_u)}{f(x_l) - f(x_u)} \rightarrow x_r = 3.3163543 \text{ and}$$

$$f(3.3163543) = 3.3163543^2 - 11 = -0.00179415$$

$$f(x_l) * f(x_r) = -0.019227 * -0.00179415$$

$$\rightarrow 0.0000344961 > 0 \text{ (root is between } x_r \text{ and } x_u)$$

SET $x_l = x_r = 3.3163543$ and CONTINUE to iterate.



4th iteration: $x_l = 3.3163543$, $x_u = 4$

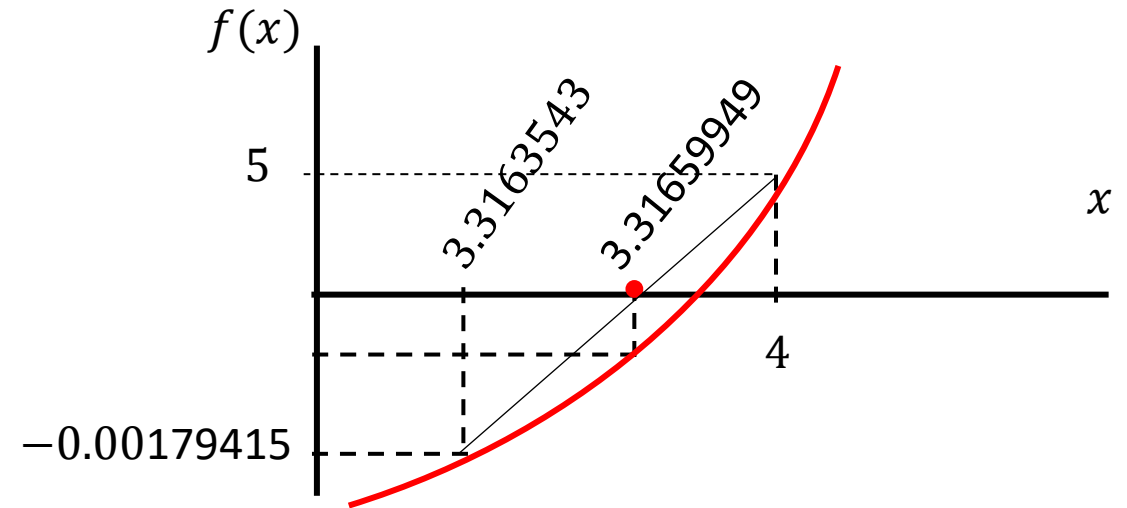
$$x_r = x_u - \frac{f(x_u)(x_l - x_u)}{f(x_l) - f(x_u)} \rightarrow x_r = 3.31659949 \text{ and}$$

$$f(3.31659949) = 3.31659949^2 - 11 = -0.0001678$$

$$f(x_l) * f(x_r) = -0.00179415 * -0.0001678$$

$$\rightarrow 0.0000344961 > 0 \text{ (root is between } x_r \text{ and } x_u)$$

SET $x_l = x_r = 3.3163543$ and CONTINUE to iterate.



Iteration	x_l	x_r	x_u	$f(x_l)$	$f(x_u)$	$f(x_r)$	$f(x_l)f(x_r)$	$ \varepsilon_a $ (%)
1	3	3.285714	4	-2	5	-0.2040816	0.4081632653	--
2	3.285714	3.313725	4	-0.20408	5	-0.0192234	0.0039231379	0.84531
3	3.313725	3.316354	4	-0.01922	5	-0.0017969	0.0000345424	0.07926
4	3.316354	3.316599	4	-0.0018	5	-0.0001678	0.0000003016	0.00741
5	3.316599	3.316622	4	-0.00017	5	-0.0000157	0.0000000026	0.00069
6	3.316622	3.316625	4	-1.6E-05	5	-0.0000015	0.0000000000	0.00006

- Note that false-position method converged faster than the bisection method.
- *Iterations between 4-6 are additional iterations.*

Notes on Bracketing Methods

- A plot of the function is always helpful.
 - To determine the number of all roots, if there are any.
 - to determine whether the roots are multiple or not.
 - to determine whether the method converges to the desired root.
 - to determine the initial guesses.
- Incremental search technique can be used to determine the initial guesses.
 - Start from one end of the region of interest.
 - Evaluate the function at specified intervals.
 - If the sign of the function changes, then there is a root in that interval.
 - Select your intervals small, otherwise you may miss some of the roots. But if they are too small, incremental search might become too costly.
 - Incremental search, just by itself, can be used as a root finding technique with very small intervals (not efficient).

Exam Question

1. (35p) To find the square root of a number ' a ', one can use the following equation:

$$f(c) = c^2 - a = 0$$

where c is the square root of a .

Use **Bisection method** to determine the square root of 3 with an absolute approximate relative percentage error (ϵ_a) below 1%. Use initial guesses of $c_l = 1$ and $c_u = 2$.

1. a) Check function changes sign: at least one root between initial lower and upper limits.
 $f(c_l)f(c_u) = f(1)f(2)(-2.000)(1.000) = -2.000 < 0$ (function changes sign, the root has to be between 1 and 2)

Step 1: Bracket [1, 2] as given

$$\begin{aligned}c_r &= \frac{c_l + c_u}{2} = \frac{1 + 2}{2} = 1.5 \\f(c_l) &= f(1) = c_l^2 - a = 1 - 3 = -2.000 \\f(c_r) &= f(1.5) = c_r^2 - a = (1.5)^2 - 3 = -0.7500 \\f(c_l)f(c_r) &= (-2.000)(-0.7500) = 1.500 > 0\end{aligned}$$

(function does not change sign, the root can not be between 1 and 1.5 s. change c_r as lower limit, upper limit will be same).

Step 2: New Bracket [1.5, 2]

$$c_r = \frac{c_l + c_u}{2} = \frac{1.5 + 2}{2} = 1.75$$

$$f(c_l) = f(1.5) = -0.75$$

$$f(c_r) = f(1.75) = 0.0625$$

$$f(c_l)f(c_r) = (-0.75)(0.0625) = -0.04688 < 0$$

(the root has to be between 1.5 and 1.75; new bracket will be [1.5, 1.75])

$$\epsilon_a = \left| \frac{1.75 - 1.5}{1.75} \right| \times 100 = 14.3\% > 1\%$$

Step 3: New Bracket [1.5, 1.75]

$$c_r = \frac{c_l + c_u}{2} = \frac{1.5 + 1.75}{2} = 1.625$$

$$f(c_l) = f(1.5) = -0.75$$

$$f(c_r) = f(1.625) = (1.625)^2 - 3 = -0.3594$$

$$f(c_l)f(c_r) = (-0.75)(-0.3594) > 0$$

(the root can't be between 1.5 and 1.625; new bracket will be [1.625, 1.75])

$$\epsilon_a = \left| \frac{1.625 - 1.75}{1.625} \right| \times 100 = 7.69\% > 1\%$$

Step 4: New Bracket [1.625, 1.75]

$$\begin{aligned}c_r &= \frac{c_l + c_u}{2} = \frac{1.625 + 1.75}{2} = 1.6875 \\f(c_l) &= f(1.625) = -0.3594 \\f(c_r) &= f(1.6875) = (1.6875)^2 - 3 = -0.1523 \\f(c_l)f(c_r) &= (-0.3594)(-0.1523) > 0\end{aligned}$$

(the root can't be between 1.625 and 1.6875; new bracket will be [1.6875, 1.75])

$$\epsilon_a = \left| \frac{1.6875 - 1.625}{1.6875} \right| \times 100 = 3.70\% > 1\%$$

Step 5: New Bracket [1.6875, 1.75]

$$\begin{aligned}c_r &= \frac{c_l + c_u}{2} = \frac{1.6875 + 1.75}{2} = 1.7188 \\f(c_l) &= f(1.6875) = -0.1523 \\f(c_r) &= f(1.7188) = (1.7188)^2 - 3 = -0.0459 \\f(c_l)f(c_r) &= (-0.1523)(-0.0459) > 0\end{aligned}$$

(the root can't be between 1.6875 and 1.7188; new bracket will be [1.7188, 1.75])

$$\epsilon_a = \left| \frac{1.7188 - 1.6875}{1.7188} \right| \times 100 = 1.8\% > 1\%$$

Step 6: New Bracket [1.7188, 1.75]

$$c_r = \frac{c_l + c_u}{2} = \frac{1.7188 + 1.75}{2} = 1.7344$$

$$f(c_l) = f(1.7188) = -0.0459$$

$$f(c_r) = f(1.7344) = (1.7344)^2 - 3 = 0.008143$$

$$f(c_l)f(c_r) = (-0.0459)(0.008143) < 0$$

(the root has to be between 1.7188 and 1.7344; new bracket will be [1.7188, 1.7344])

$$\epsilon_a = \left| \frac{1.7344 - 1.7188}{1.7344} \right| \times 100 = 0.90\% < 1\%$$

Thus after 6 iterations, the approximate error falls below 1% with a result of $c_r = 1.7344$ as tabulated below:

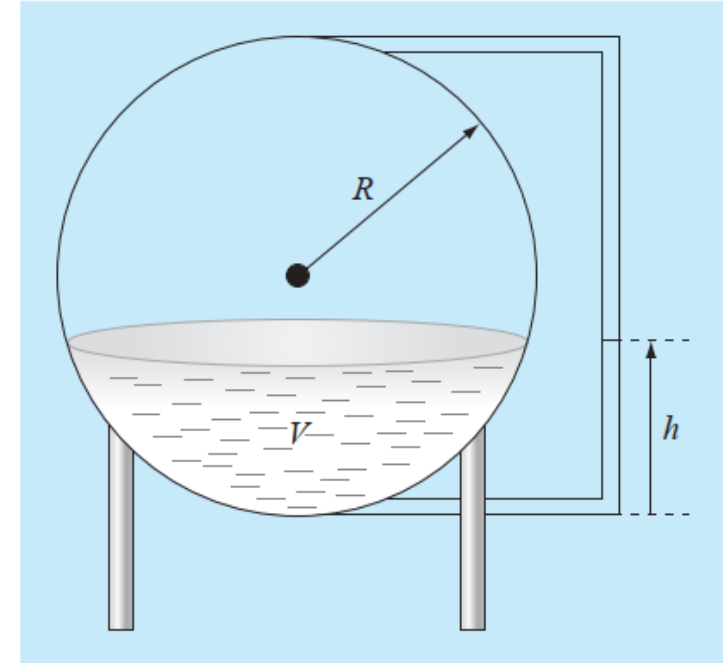
i	c_l	c_u	c_r	$f(c_l)$	$f(c_r)$	$f(c_l)f(c_r)$	$\epsilon_a, \%$
1	1.0000	2.0000	1.5000	-2.000	-0.7500	>0	
2	1.5000	2.0000	1.7500	-0.7500	0.0625	<0	14.30
3	1.5000	1.7500	1.6250	-0.7500	-0.3594	>0	7.69
4	1.6250	1.7500	1.6875	-0.3594	-0.1523	>0	3.70
5	1.6875	1.7500	1.7188	-0.1523	-0.0459	>0	1.80
6	1.7188	1.7500	1.7344	-0.0459	0.008143	<0	0.90

Problem 5.17

You are designing a spherical tank to hold water for a small village in a developing country. The volume of liquid it can hold can be computed as

$$V = \pi h^2 \frac{[3R - h]}{3}$$

where V = volume [m^3], h = depth of water in tank [m], and R = the tank radius [m]. If $R = 3\text{ m}$, to what depth must the tank be filled so that it holds 30 m^3 ? Use three iterations of the false-position method to determine your answer. Determine the approximate relative error after each iteration. Employ initial guesses of 0 and R .



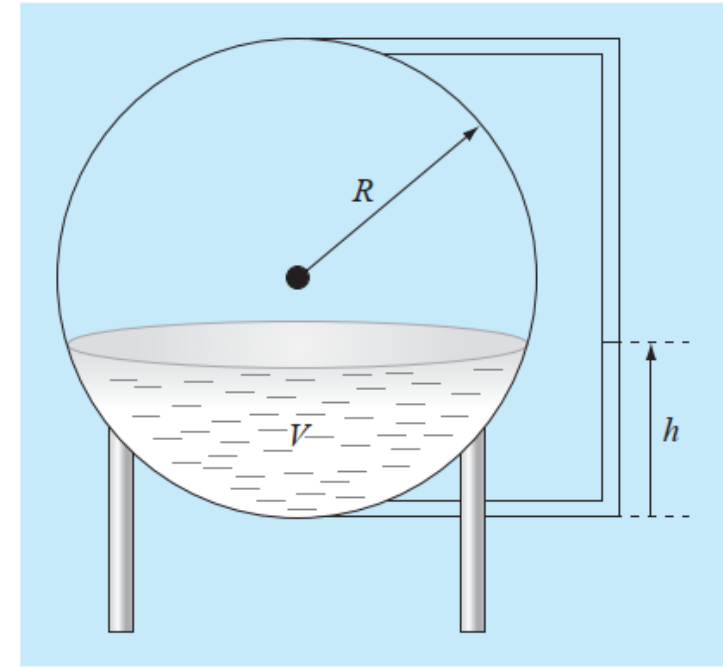
To find the root (h) of the function $f(h)$, we need to represent the function as $f(h) = 0$

$$V = \pi h^2 \frac{[3R - h]}{3} \quad \begin{cases} V = 30 \text{ m}^3 \\ R = 3 \text{ m} \\ h = ? \end{cases}$$

$$f(h) = 30 = \frac{\pi h^2 (9 - h)}{3} \rightarrow f(h) = \frac{\pi h^2 (9 - h)}{3} - 30 = 0$$

- Initial estimates of h are $h_l = 0$ and $h_u = 3$, where the function is continuous at both points.

$f(0) = -30$ and $f(3) \approx 26.5487$ since $f(h_l) * f(h_u) < 0$ the initial estimates are useful.

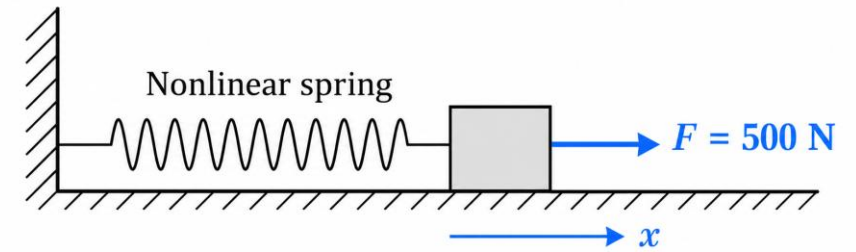


Iteration	(h_l)	(h_u)	$(h_r) (m)$	$(f(h_r)) (m^3)$	(ε_a)
1	0	3	1.591549	-10.348475	—
2	1.591549	3	1.986575	-1.015307	%19.8848
3	1.986575	3	2.023904	-0.075913	%1.8444

Question

We performed three iterations. Does this mean we have achieved the desired accuracy?

Homework: Equilibrium of a Nonlinear Spring



The restoring force of a nonlinear spring is

$$F_s(x) = k_1 x + k_3 x^3$$

$$F_s(x) = k_1 x + k_3 x^3$$

$$k_1 = 1000 \text{ N/m} \quad k_3 = 10000 \text{ N/m}^3$$

where $k_1 = 1000 \text{ N/m}$, $k_3 = 10000 \text{ N/m}^3$, and x is the displacement in metres. An external force of $F = 500 \text{ N}$ is applied.

1. Formulate the equilibrium condition as a root-finding problem.
2. Verify that $[0, 0.5] \text{ m}$ brackets a root.
3. Solve using both bisection and false-position. Stop when the absolute approximate relative percentage error is below 0.1%.
4. For each method, report i , x_l , x_u , x_r , $f(x_r)$, and ε_a .
5. Compare the iteration counts and the final force residuals. Does the same stopping tolerance produce identical results?
6. Retain full precision during calculations and report six decimal places. You may use a spreadsheet or a short program.

NEXT WEEK
ROOTS OF EQUATIONS
Open Methods