ME 209 NUMERICAL METHODS
FINAL EXAM

Name Surname: **05.01.2026**
Student No: **Duration: 100 min**

Q1. (25 p) The Beattie-Bridgeman equation of state

$$P = \frac{RT}{V} + \frac{a}{V^2} + \frac{b}{V^3} + \frac{c}{V^4}$$

is a three-parameter extension of the ideal gas law. Using $\boldsymbol{a = -1.06, b = 0.057, and\ c = -0.0001}$ find the **<u>volume of 1 mole of a gas</u>** at $P = 25\ atm$, and T = 293 K. The constant R = *0.082 L.atm/(mol.K)* is the ideal gas constant in °C. Use **<u>Bisection method</u>** to with $\boldsymbol{V_l}$**= 0.9 *L/mol***, $\boldsymbol{V_u}$ **= 1 *L/mol*** with a specified absolute approximate relative percentage error $\boldsymbol{\varepsilon_s} = $ **3%**.

Q2. (25 p) Use the Gauss-Seidel method to solve the following system until the percent relative error falls below ε_s = 5%. Use initial guesses of unknowns $x_1 = 0$; $x_2 = 0$; $x_3 = 0$

$$2x_1 - 6x_2 - x_3 = -38$$

$$-8x_1 + x_2 - 2x_3 = -20$$

$$-3x_1 - x_2 + 7x_3 = -34$$

Q3. (25 p) Evaluate the following integral by **<u>composite Simpson's 1/3 Rule with n= 8 segments</u>**.

$$\int_2^6 \frac{1}{\sqrt{(2x-1)}}\,dx$$

Q4. (25 p) The relationship between the voltage applied to an electrical circuit and the current flowing is as shown:

Current (*mA*)	2	4	6	8	10	12	14
Applied Voltage (*V*)	5	11	15	19	24	28	33

Assuming a linear relationship, determine the equation of the regression line of applied voltage ***Y*** on the current ***X*** correct to <u>3 decimal places</u>. ($Y = a_0 + a_1 X$)

$$I \cong (b-a)\frac{f(x_0) + 4\sum_{i=1,3,5}^{n-1} f(x_i) + 2\sum_{j=2,4,6}^{n-2} f(x_j) + f(x_n)}{3n}$$

$$a_0 = \bar{y} - a_1\bar{x} \qquad a_1 = \frac{n\Sigma x_i y_i - \Sigma x_i \Sigma y_i}{n\Sigma x_i^2 - (\Sigma x_i)^2}$$

$\bar{y}$ and $\bar{x}$ are the means of y and x, respectively.

GOOD LUCK.

1.

By substituting the values given in the Beattie-Bridgeman equation of state, the solution can be formulated as follows:

$$25 = \frac{0.082(293)}{V} + \frac{(-1.06)}{V^2} + \frac{0.057}{V^3} + \frac{(-0.0001)}{V^4}$$

$$\rightarrow f(V) = \frac{24.026}{V} - \frac{1.06}{V^2} + \frac{0.057}{V^3} - \frac{1x10^{-4}}{V^4} - 25 = 0$$

Applying Bisection method with the initial guesses of $V_l = 0.9\ L/mol\,; V_u = 1\ L/mol$ gives:

$$Check: f(V_l)f(V_u) = f(0.9)f(1) = (0.46495)(-1.9771) < 0$$

Iteration 1: Bracket $[0.9; 1]$

$$V_r = \frac{V_l + V_u}{2} = \frac{0.9 + 1}{2} = 0.95$$

$$f(V_l)f(V_r) = f(0.9)f(0.95) = (0.46495)(-0.8176) < 0$$

The root should be between $V_l = 0.9;\ V_r = V_u = 0.95.$

Iteration 2: Bracket$[0.9; 0.95]$

$$V_r = \frac{V_l + V_u}{2} = \frac{0.9 + 0.95}{2} = 0.925$$

$$\varepsilon_a = \left|\frac{0.925 - 0.95}{0.925}\right| \times 100 = 2.7\% < \varepsilon_s\ (3\%)\ STOP$$

$$Root\ \mathbf{V} = \mathbf{0.925}\ with\ an\ absolute\ relative\ approximate\ error\ of\ 2.7\%$$

Note: If continued, Iteration 3 bracket will be : $[0.9; 0.925]$, $because\ f(V_l)f(V_r) = f(0.9)f(0.925) = (0.46495)(-0.1929) < 0, V_r = 0.9125, \varepsilon_a = 1.37\%$

2. First, the equation system should be pivoted in order to obtain coefficients with higher absolute value in the diagonal elements. On the other hand, solution will not converge to the real roots.

$$-8x_1 + x_2 - 2x_3 = -20$$
$$2x_1 - 6x_2 - x_3 = -38$$
$$-3x_1 - x_2 + 7x_3 = -34$$

Iteration 0:

$$x_1 = 0; x_2 = 0; x_3 = 0$$

Iteration 1:

$$x_1 = \frac{-20 - x_2 + 2x_3}{-8} = \frac{-20 - 0 + 2(0)}{-8} = 2.5$$

$$x_2 = \frac{-38 - 2x_1 + x_3}{-6} = \frac{-38 - 2(2.5) + 0}{-6} = 7.166667$$

$$x_3 = \frac{-34 + 3x_1 + x_2}{7} = \frac{-34 + 3(2.5) + 7.166667}{7} = -2.761905$$

Iteration 2:

$$x_1 = \frac{-20 - x_2 + 2x_3}{-8} = \frac{-20 - 7.166667 + 2(-2.761905)}{-8} = 4.08631$$

$$x_2 = \frac{-38 - 2x_1 + x_3}{-6} = \frac{-38 - 2(4.08631) + (-2.761905)}{-6} = 8.155754$$

$$x_3 = \frac{-34 + 3x_1 + x_2}{7} = \frac{-34 + 3(4.08631) + 8.155754}{7} = -1.94076$$

The error estimates can be computed as follows:

$$\varepsilon_{a,1} = \left|\frac{4.08631 - 2.5}{4.08631}\right| \times 100 = 38.82\%$$

$$\varepsilon_{a,2} = \left|\frac{8.155754 - 7.166667}{8.155754}\right| \times 100 = 12.13\%$$

$$\varepsilon_{a,3} = \left|\frac{-1.94076 - (-2.761905)}{-1.94076}\right| \times 100 = 42.31\%$$

Iteration 3:

$$x_1 = \frac{-20 - x_2 + 2x_3}{-8} = \frac{-20 - 8.155754 + 2(-1.94076)}{-8} = 4.004659$$

$$x_2 = \frac{-38 - 2x_1 + x_3}{-6} = \frac{-38 - 2(4.004659) + (-1.94076)}{-6} = 7.99168$$

$$x_3 = \frac{-34 + 3x_1 + x_2}{7} = \frac{-34 + 3(4.004659) + 7.99168}{7} = -1.99919$$

The error estimates can be computed as follows:

$$\varepsilon_{a,1} = \left|\frac{4.004659 - 4.08631}{4.004659}\right| \times 100 = 2.04\%$$

$$\varepsilon_{a,2} = \left|\frac{7.99168 - 8.155754}{7.99168}\right| \times 100 = 2.05\%$$

$$\varepsilon_{a,3} = \left|\frac{-1.99919 - (-1.94076)}{-1.99919}\right| \times 100 = 2.92\% \ (maximum\ error) < 5\%$$

The solution vector is:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 4.04659 \\ 7.99168 \\ -1.99919 \end{bmatrix}$$

3.

b) Applying Composite Simpson's 1/3 Rule:

$$I \cong (b-a)\frac{f(x_0) + 4\sum_{i=1,3,5}^{n-1} f(x_i) + 2\sum_{j=2,4,6}^{n-2} f(x_j) + f(x_n)}{3n}$$

$$a = x_0 = 2, b = x_n = x_8 = 6, n = 8\ (given), h = \frac{b-a}{n} = \frac{6-2}{8} = 0.5, x_1 = 2.5, x_2 = 3, x_3 = 3.5, x_4 = 4, x_5 = 4.5, x_6 = 5, x_7 = 5.5, x_8 = x_n = 6$$

$$I \cong (6-2)\frac{f(2) + 4[f(2.5) + f(3.5) + f(4.5) + f(5.5)] + 2[f(3) + f(4) + f(5)] + f(6)}{24}$$

$$I \cong (6-2)\frac{0.5774 + 4[0.5 + 0.4082 + 0.3536 + 0.3162] + 2[0.4472 + 0.3780 + 0.3333] + 0.3015}{24}$$

$$= 1.585\ correct\ to\ 3\ decimal\ places$$

4.

Table. Summations of data to calculate coefficients of the linear model

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i	X_i	Y_i	X_iY_i	X_i^2
1	2	5	10	4
2	4	11	44	16
3	6	15	90	36
4	8	19	152	64
5	10	24	240	100
6	12	28	336	144
7	14	33	462	196
Σ	56	135	1334	560

By substituting the corresponding values at the bottom of the table to the least square equations given above, we have:

$$a_1 = \frac{7(1334) - (56)(135)}{7(560) - (56)^2} = 2.268$$

$$a_0 = \frac{135}{7} - 2.268\frac{56}{7} = 1.142$$

Linear model to fit the data:

$$Y = 1.142 + 2.268X$$