# ME 209
# Numerical Methods

## 6. Interpolation Methods 1

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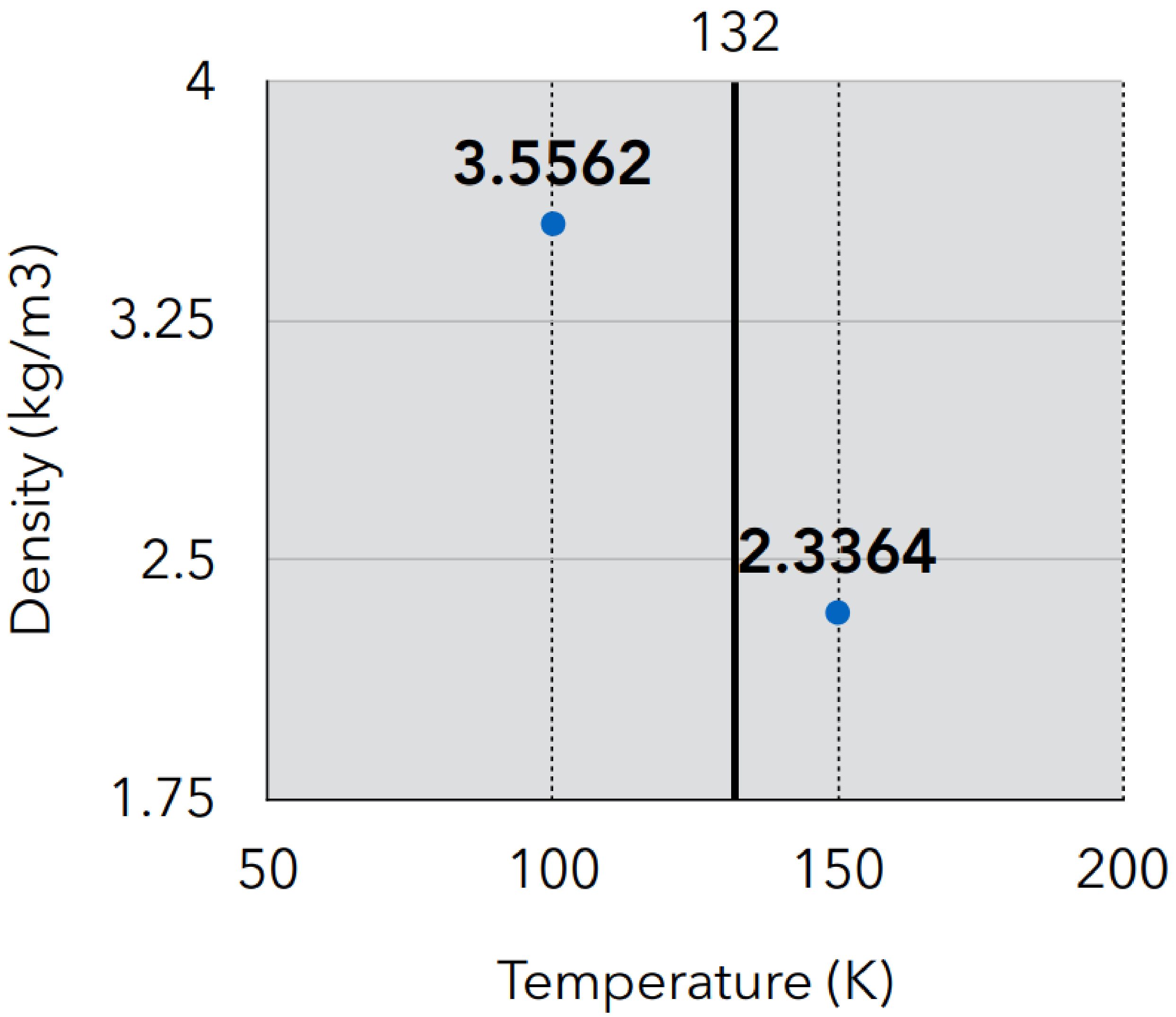
- Motivation:
  - Often we have discrete data (tabulated, from experiments, etc) that we need to interpolate.
  - Interpolating functions form the basis for numerical integration and differentiation techniques
    - Used for solving ODEs & PDEs
    - we will cover this later
- Concept:
  - Choose a polynomial function to fit to the data (connect the dots)
  - Solve for the coefficients of the polynomial
  - Evaluate the polynomial wherever you want (interpolation)

| T<br>K | ρ<br>kg/m³ | λ<br>W/(m K) | μ<br>N s/m² |
|---|---|---|---|
| 100 | 3.5562 | 0.0093 | 7.110e-06 |
| 150 | 2.3364 | 0.0138 | 1.034e-05 |
| 200 | 1.7458 | 0.0181 | 1.325e-05 |
| 250 | 1.3947 | 0.0223 | 1.596e-05 |
| 300 | 1.1614 | 0.0263 | 1.846e-05 |
| 350 | 0.9950 | 0.0300 | 2.082e-05 |
| 400 | 0.8711 | 0.0338 | 2.301e-05 |
| 450 | 0.7750 | 0.0373 | 2.507e-05 |
| 500 | 0.6864 | 0.0407 | 2.701e-05 |
| 550 | 0.6329 | 0.0439 | 2.884e-05 |
| 600 | 0.5804 | 0.0469 | 3.058e-05 |
| 650 | 0.5356 | 0.0497 | 3.225e-05 |
| 700 | 0.4975 | 0.0524 | 3.388e-05 |
| 750 | 0.4643 | 0.0549 | 3.546e-05 |
| 800 | 0.4354 | 0.0573 | 3.698e-05 |
| 850 | 0.4097 | 0.0596 | 3.843e-05 |
| 900 | 0.3868 | 0.0620 | 3.981e-05 |
| 950 | 0.3666 | 0.0643 | 4.113e-05 |
| 1000 | 0.3482 | 0.0667 | 4.244e-05 |

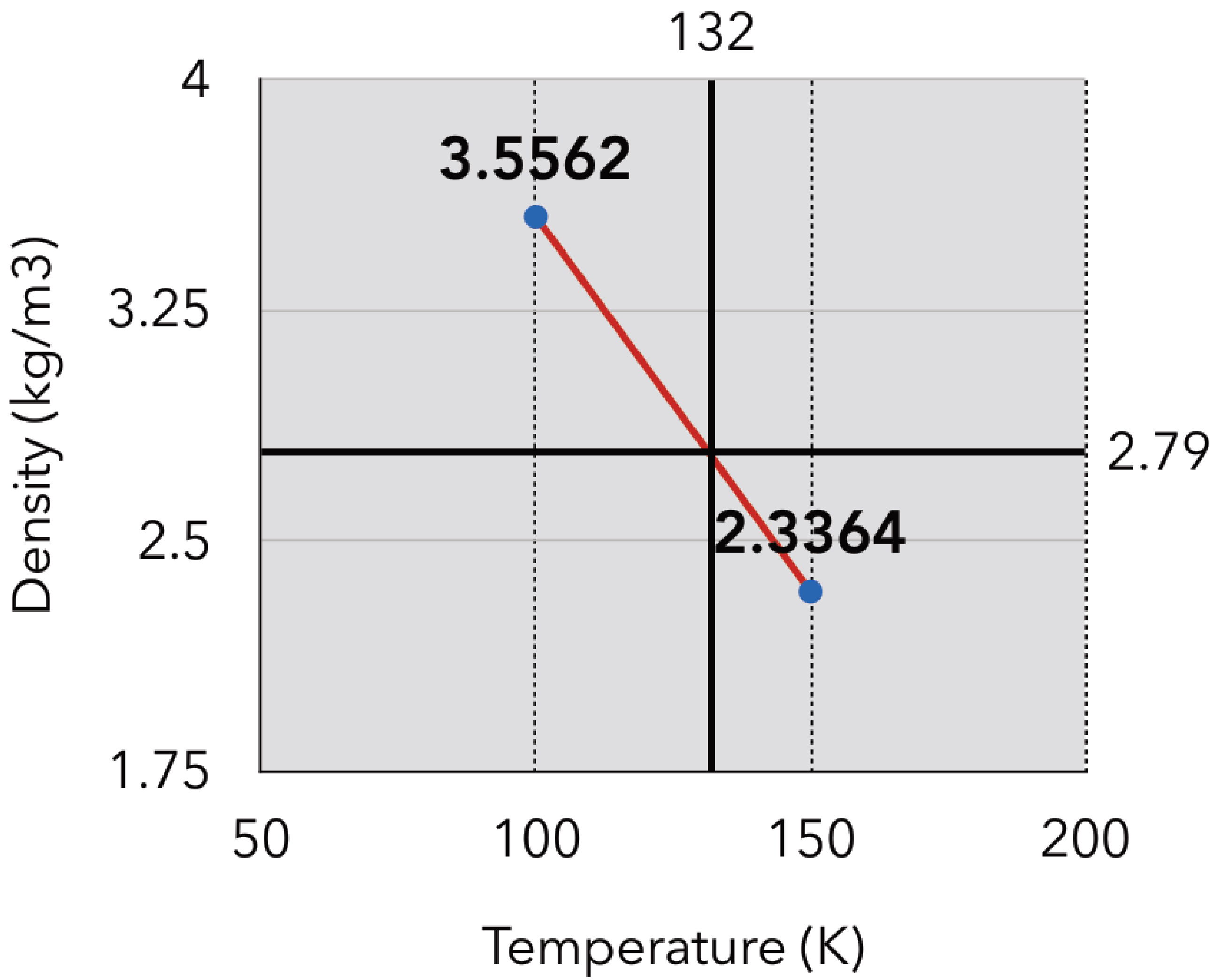
Incropera & DeWitt, *Fundamentals of Heat and Mass Transfer*, 4th ed.




Properties of air at atmospheric pressure

| T | ρ |
|---|---|
| K | $kg/m^3$ |
| 100 | 3.5562 |
| 150 | 2.3364 |

find density @ T = 132 K




Properties of air at atmospheric pressure

| T K | ρ kg/m³ |
|---|---|
| 100 | 3.5562 |
| 150 | 2.3364 |

find density @ T = 132 K

Find slope: $s = \dfrac{2.34 - 3.56}{50} = -0.024$

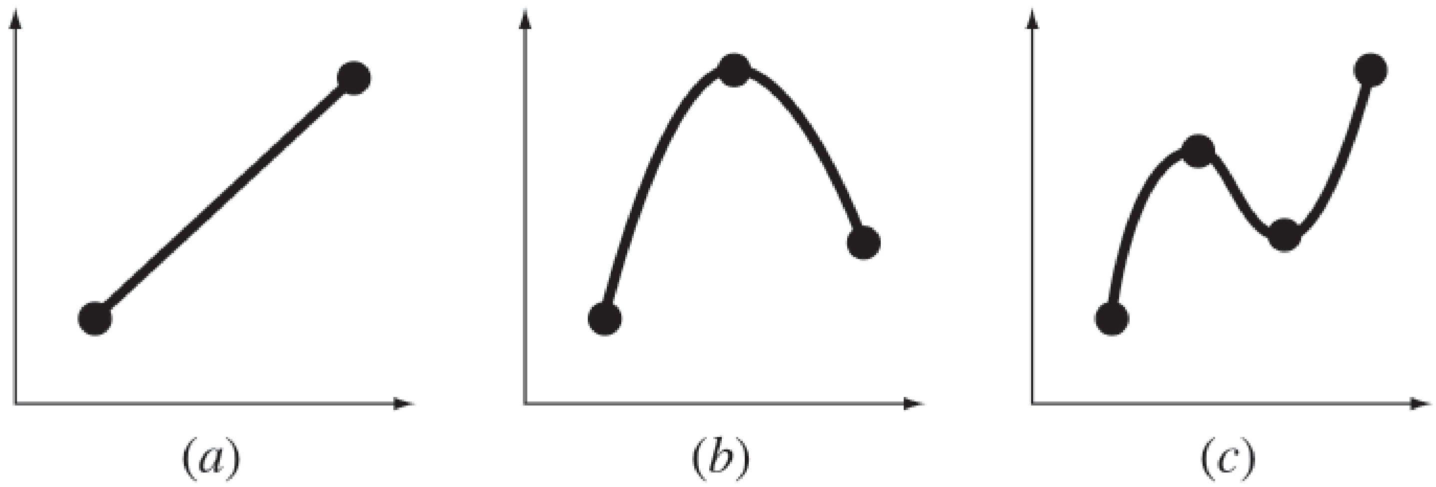
Find value:

Connect via straight line

$$\rho(132) = 3.56 + s \times (132 - 100) = 2.79$$

# INTRODUCTION

- *Interpolation* is a method of estimating the intermediate values between precise data points. The most common method used for this purpose is polynomial interpolation.
- The basis of all interpolation algorithms is the fitting of some type of curve or function to a subset of the tabular data.
- Thus, we first fit a function that exactly passes through the given data points and than evaluate intermediate values using this function.



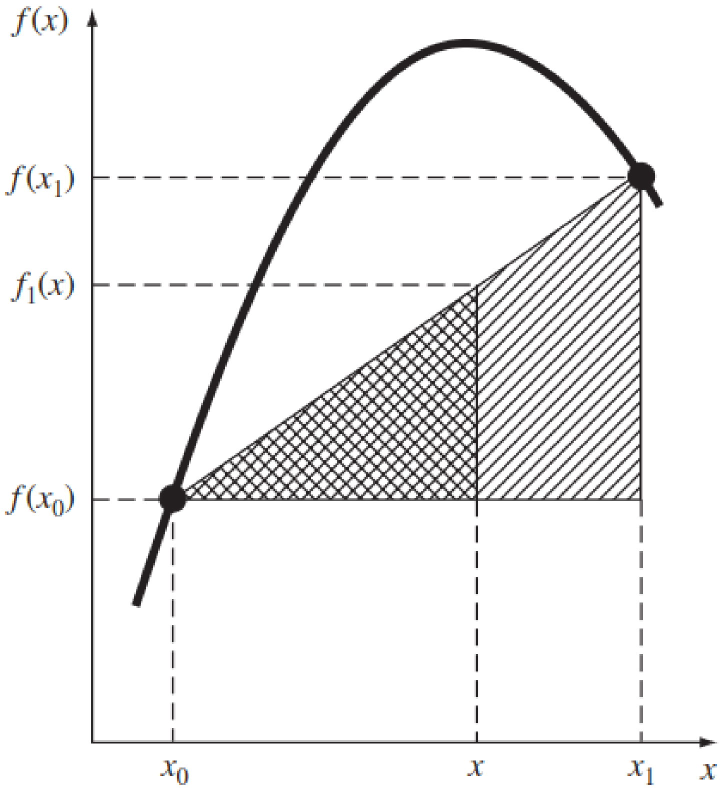

a) first-order (linear) connecting two points,
b) second-order (quadratic or parabolic) connecting three points, and
c) third-order (cubic) connecting four points.

# NEWTON’S DIVIDED-DIFFERENCE INTERPOLATING POLYNOMIALS

There are a variety of alternative forms for expressing an interpolating polynomial. *Newton’s divided-difference interpolating polynomial* is among the most popular and useful forms. Before presenting the general equation, we will introduce the first- and second-order versions

# Linear Interpolation

The simplest form of interpolation is to connect two data points with a straight line. This technique, called *linear interpolation,*




From similar triangles,

$$\frac{f_1(x) - f(x_0)}{x - x_0} = \frac{f(x_1) - f(x_0)}{x_1 - x_0}$$

*linear-interpolation formula*

$$\boxed{f_1(x) = f(x_0) + \frac{f(x_1) - f(x_0)}{x_1 - x_0}(x - x_0)}$$

The slope of the line connecting the points, the term $[f(x_1) - f(x_0)]/(x_1 - x_0)$ is a finite-divided-difference approximation of the first derivative

### Linear Interpolation

Problem Statement. Estimate the natural logarithm of 2 using linear interpolation. First, perform the computation by interpolating between ln 1 = 0 and ln 6 = 1.791759. Then, repeat the procedure, but use a smaller interval from ln 1 to ln 4 (1.386294). Note that the true value of ln 2 is 0.6931472.

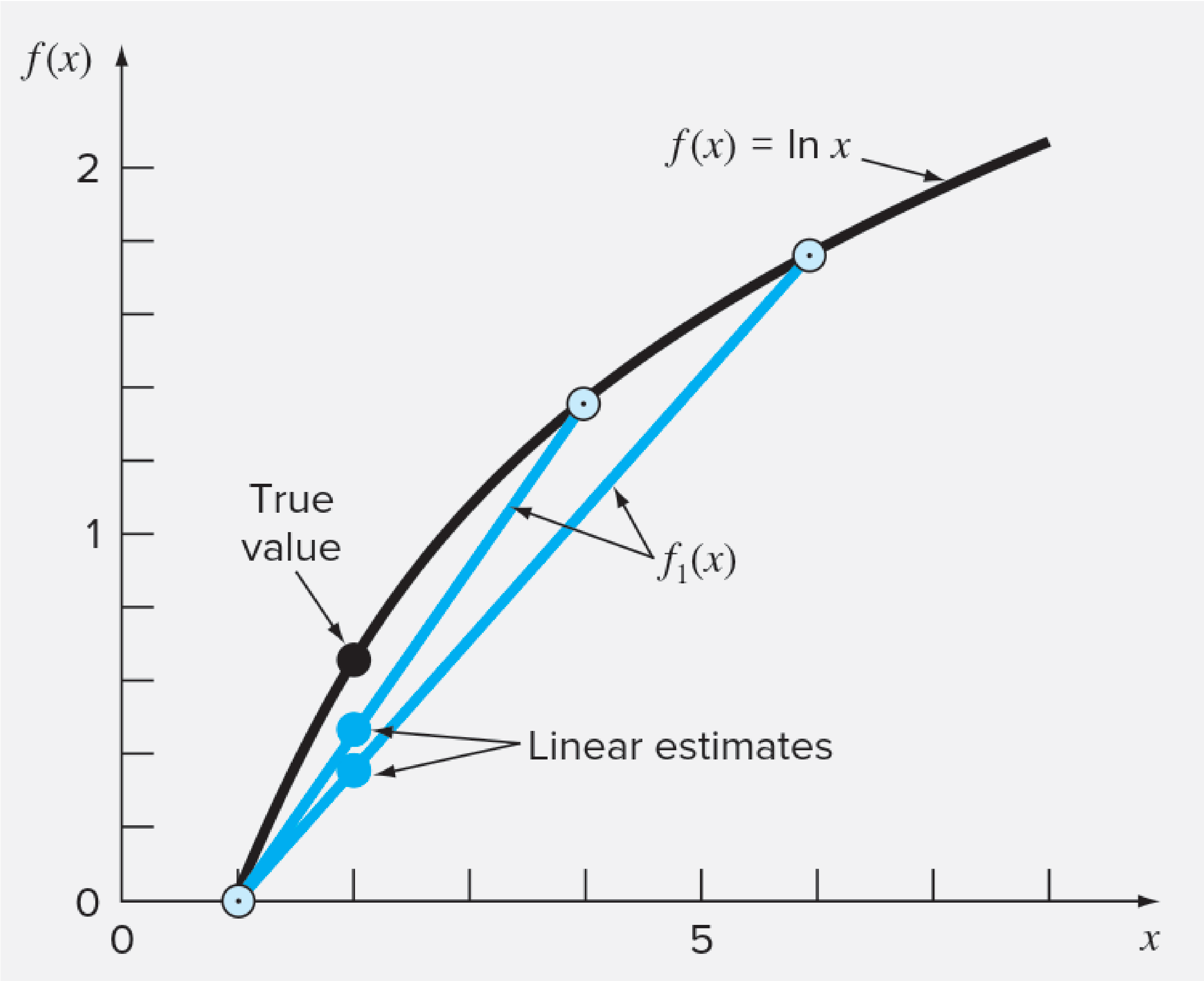
Solution. We use Eq. (18.2) and a linear interpolation for ln 2 from $x_0 = 1$ to $x_1 = 6$ to give

$$f_1(2) = 0 + \frac{1.791759 - 0}{6 - 1}(2 - 1) = 0.3583519$$

which represents an error of $\varepsilon_t = 48.3\%$. Using the smaller interval from $x_0 = 1$ to $x_1 = 4$ yields

$$f_1(2) = 0 + \frac{1.386294 - 0}{4 - 1}(2 - 1) = 0.4620981$$

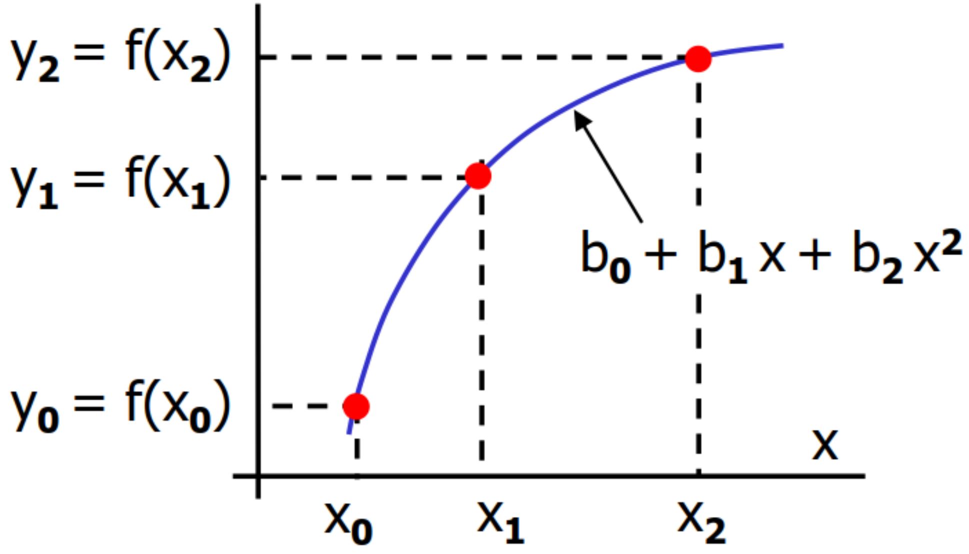
Thus, using the shorter interval reduces the percent relative error to $\varepsilon_t = 33.3\%$. Both



f(x)
2
1
0
0
5
x
f(x) = ln x
True
value
f_1(x)
Linear estimates

# Quadratic Interpolation

If three data points are available to find desired point, this can be accomplished with a *second-order polynomial* (also called a quadratic polynomial, or a *parabola*).




- Given: $(x_0, y_0)$ , $(x_1, y_1)$ and $(x_2, y_2)$
- A parabola passes from these three points.
- Similar to the linear case, the equation of this parabola can be written as

$$\boxed{f_2(x) = b_0 + b_1(x - x_0) + b_2(x - x_0)(x - x_1)}$$

A simple procedure can be used to determine the values of the coefficients. For $b^0$, Eq. (18.3) with $x = x^0$ can be used to compute

# Quadratic Interpolation

$$\boxed{f_2(x) = b_0 + b_1(x - x_0) + b_2(x - x_0)(x - x_1)}$$

A simple procedure can be used to determine the values of the coefficients.

For $b_0$, $x = x^0$ can be used to compute

$$b_0 = f(x_0)$$

at $x = x_1$

$$b_1 = \frac{f(x_1) - f(x_0)}{x_1 - x_0}$$

at $x = x_2$

$$b_2 = \frac{\dfrac{f(x_2) - f(x_1)}{x_2 - x_1} - \dfrac{f(x_1) - f(x_0)}{x_1 - x_0}}{x_2 - x_0}$$

## Quadratic Interpolation

Problem Statement. Fit a second-order polynomial to the three points used in Example 18.1:

$$
\begin{aligned}
x_0 &= 1 \qquad & f(x_0) &= 0 \\
x_1 &= 4 \qquad & f(x_1) &= 1.386294 \\
x_2 &= 6 \qquad & f(x_2) &= 1.791759
\end{aligned}
$$

Use the polynomial to evaluate ln 2.

**Solution.** Applying Eq. (18.4) yields

$$b_0 = 0$$

Equation (18.5) yields

$$b_1 = \frac{1.386294 - 0}{4 - 1} = 0.4620981$$

and Eq. (18.6) gives

$$b_2 = \frac{\dfrac{1.791759 - 1.386294}{6 - 4} - 0.4620981}{6 - 1} = -0.0518731$$

$$f_2(x) = 0 + 0.4620981(x - 1) - 0.0518731(x - 1)(x - 4)$$

which can be evaluated at $x = 2$ to give

$$f_2(2) = 0.5658444$$

which represents a relative error of $\varepsilon_t = 18.4\%$.

# General Form of Newton's Interpolating Polynomials

The preceding analysis can be generalized to fit an *n*th-order polynomial to $n + 1$ data points. The *n*th-order polynomial is

$$f_n(x) = b_0 + b_1(x - x_0) + \cdots + b_n(x - x_0)(x - x_1) \cdots (x - x_{n-1})$$

$$b_0 = f(x_0)$$

$$b_1 = f[x_1, x_0]$$

$$b_2 = f[x_2, x_1, x_0]$$

$$\vdots$$

$$b_n = f[x_n, x_{n-1}, \ldots, x_1, x_0]$$

where the bracketed function evaluations are finite divided differences.

first finite divided difference

$$f[x_i, x_j] = \frac{f(x_i) - f(x_j)}{x_i - x_j}$$

second finite divided difference

$$f[x_i, x_j, x_k] = \frac{f[x_i, x_j] - f[x_j, x_k]}{x_i - x_k}$$

$n^{th}$ finite divided difference

$$f[x_n, x_{n-1}, \ldots, x_1, x_0] = \frac{f[x_n, x_{n-1}, \ldots, x_1] - f[x_{n-1}, x_{n-2}, \ldots, x_0]}{x_n - x_0}$$

These differences can be used to evaluate the coefficients *b*

*Newton's divided-difference interpolating polynomial.*

$$\boxed{\begin{aligned} f_n(x) = f(x_0) &+ (x - x_0)f[x_1, x_0] + (x - x_0)(x - x_1)f[x_2, x_1, x_0] \\ &+ \cdots + (x - x_0)(x - x_1)\cdots(x - x_{n-1})f[x_n, x_{n-1}, \ldots, x_0] \end{aligned}}$$

Graphical depiction of the recursive nature of finite divided differences.

| ***i*** | $x_i$ | $f(x_i)$ | **First** | **Second** | **Third** |
|---|---|---|---|---|---|
| 0 | $x_0$ | $f(x_0)$ | $f[x_1, x_0]$ | $f[x_2, x_1, x_0]$ | $f[x_3, x_2, x_1, x_0]$ |
| 1 | $x_1$ | $f(x_1)$ | $f[x_2, x_1]$ | $f[x_3, x_2, x_1]$ | |
| 2 | $x_2$ | $f(x_2)$ | $f[x_3, x_2]$ | | |
| 3 | $x_3$ | $f(x_3)$ | | | |

Problem Statement. data points at $x_0 = 1$, $x_1 = 4$, and $x_2 = 6$ were used to estimate ln 2 with a parabola. Now, adding a fourth point, $[x_3 = 5; f(x_3) = 1.609438]$, estimate ln 2 with a third-order Newton's interpolating polynomial.

Solution. The third-order polynomial, with $n = 3$, is

$$f_3(x) = b_0 + b_1(x - x_0) + b_2(x - x_0)(x - x_1) + b_3(x - x_0)(x - x_1)(x - x_2)$$

The first finite divided differences for the problem are

$$f[x_1, x_0] = \frac{1.386294 - 0}{4 - 1} = 0.4620981$$

$$f[x_2, x_1] = \frac{1.791759 - 1.386294}{6 - 4} = 0.2027326$$

$$f[x_3, x_2] = \frac{1.609438 - 1.791759}{5 - 6} = 0.1823216$$

The second finite divided differences are

$$f[x_2, x_1, x_0] = \frac{0.2027326 - 0.4620981}{6 - 1} = -0.05187311$$

$$f[x_3, x_2, x_1] = \frac{0.1823216 - 0.2027326}{5 - 4} = -0.02041100$$

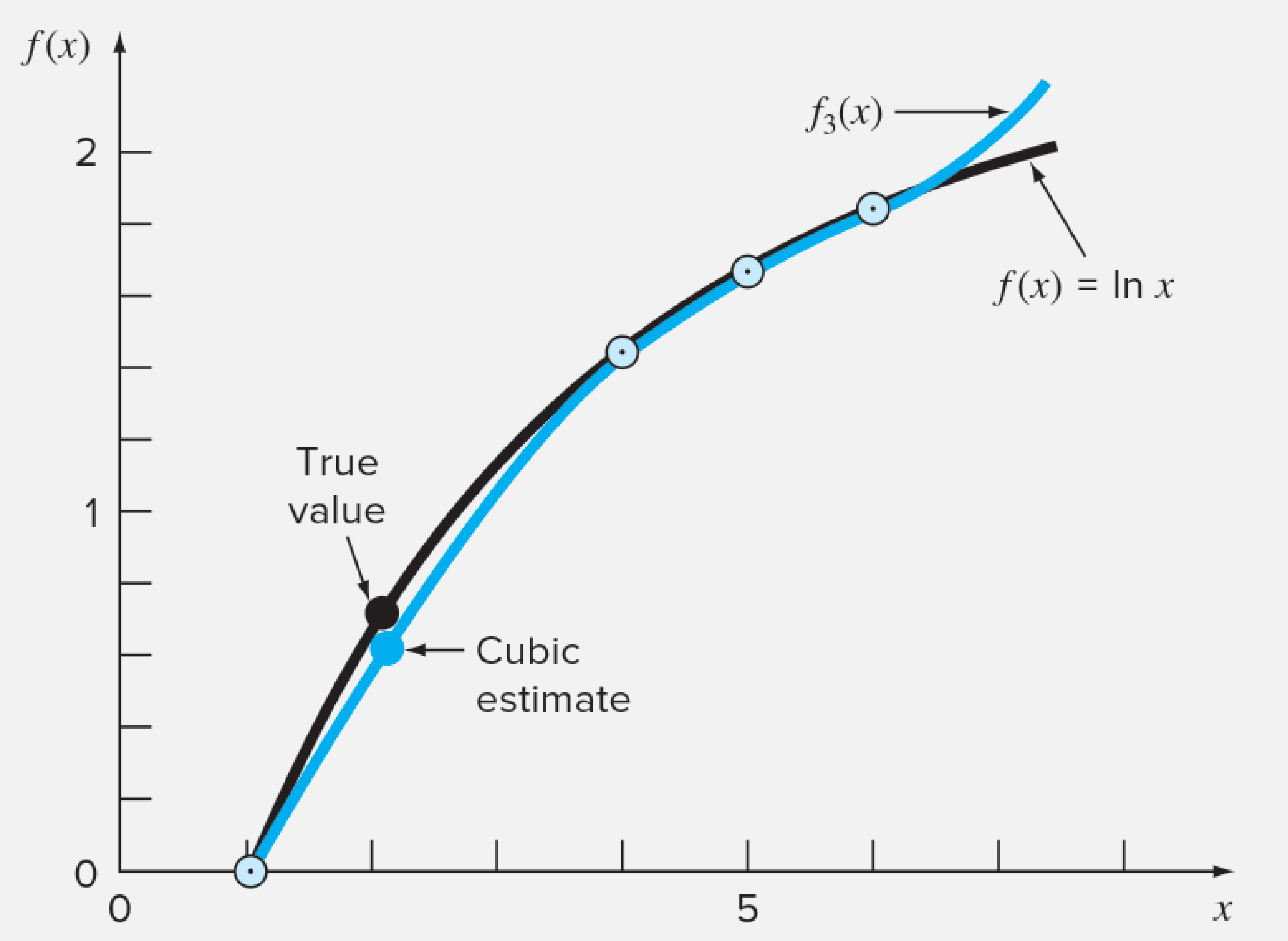
The third finite divided difference is with $n = 3$]

$$f[x_3, x_2, x_1, x_0] = \frac{-0.02041100 - (-0.05187311)}{5 - 1} = 0.007865529$$

The results for $f[x_1, x_0]$, $f[x_2, x_1, x_0]$, and $f[x_3, x_2, x_1, x_0]$ represent the coefficients $b_1$, $b_2$, and $b_3$, respectively. With $b_0 = f(x_0) = 0.0$,

$$f_3(x) = 0 + 0.4620981(x - 1) - 0.05187311(x - 1)(x - 4) + 0.007865529(x - 1)(x - 4)(x - 6)$$

which can be used to evaluate $f_3(2) = 0.6287686$, which represents a relative error of $\varepsilon_t = 9.3\%$.


$f(x)$
$f_3(x)$
2
1
0
0
5
$x$
$f(x) = \ln x$
True value
Cubic estimate

The following logarithmic table is given.

| x | f(x)=log(x) |
|---|---|
| 4.0 | 0.60206 |
| 4.5 | 0.6532125 |
| 5.5 | 0.7403627 |
| 6.0 | 0.7781513 |

(a) Interpolate log(5) using the points x=4 and x=6

(b) Interpolate log(5) using the points x=4.5 and x=5.5

Note that the exact value is log(5) = 0.69897

(a) Linear interpolation. $f(x) = f(x_0) + (x - x_0)\, f[x_1, x_0]$

$x_0 = 4,\ x_1 = 6 \ \rightarrow\ f[x_1, x_0] = [f(6) - f(4)] / (6 - 4) = 0.0880046$

$f(5) \approx f(4) + (5 - 4)\, 0.0880046 = 0.690106 \qquad \varepsilon_t = 1.27\ \%$

(b) Again linear interpolation. But this time

$x_0 = 4.5,\ x_1 = 5.5 \ \rightarrow\ f[x_1, x_0] = [f(5.5) - f(4.5)] / (5.5 - 4.5) = 0.0871502$

$f(5) \approx f(4.5) + (5 - 4.5)\, 0.0871502 = 0.696788 \qquad \varepsilon_t = 0.3\ \%$

(c) Interpolate log(5) using the points x=4.5, x=5.5 and x=6

(c) Quadratic interpolation.

$x_0 = 4.5$, $x_1 = 5.5$ , $x_2 = 6 \rightarrow$ $f[x_1, x_0] = 0.0871502$ (already calculated)

$f[x_2, x_1] = [f(6) - f(5.5)] / (6 - 5.5) = 0.0755772$

$f[x_2, x_1, x_0] = \{f[x_2, x_1] - f[x_1, x_0]\} / (6 - 4.5) = -0.0077153$

$f(5) \approx 0.696788 + (5 - 4.5)(5 - 5.5)(-0.0077153) = 0.698717$ $\varepsilon_t = 0.04\ \%$

- Note that 0.696788 was calculate in part (b).
- Errors decrease when the points used are closer to the interpolated point.
- Errors decrease as the degree of the interpolating polynomial increases.

# Finite Divided Difference (FDD) Table

Finite divided differences used in the Newton's Interpolating Polynomials can be presented in a table form. This makes the calculations much simpler.

| x | f( ) | f [ , ] | f [ , , ] | f [ , , , ] |
|---|---|---|---|---|
| $x_0$ | $f(x_0)$ | $f[x_1, x_0]$ | $f[x_2, x_1, x_0]$ | $f[x_3, x_2, x_1, x_0]$ |
| $x_1$ | $f(x_1)$ | $f[x_2, x_1]$ | $f[x_3, x_2, x_1]$ | |
| $x_2$ | $f(x_2)$ | $f[x_3, x_2]$ | | |
| $x_3$ | $f(x_3)$ | | | |

| x | f( ) | f [ , ] | f [ , , ] | f [ , , , ] |
|---|---|---|---|---|
| 4 | 0.6020600 | 0.1023050 | -0.0101032 | 0.001194 |
| 4.5 | 0.6532125 | 0.0871502 | -0.0077153 | |
| 5.5 | 0.7403627 | 0.0755772 | | |
| 6 | 0.7781513 | | | |

Use this previously calculated table to interpolate for log(5).

(a) Using points x=4 and x=4.5.

(b) Using points x=4.5 and x=5.5.

(c) Using points x=4 and x=6.

(d) Using points x=4.5 , x=5.5 and x=6.

(e) Using all four points.

(a) Using points x=4 and x=4.5.

$\log(5) \approx 0.60206 + (5 - 4)\ 0.102305 = 0.704365 \qquad \varepsilon_t = 0.8\ \%$ (this is extrapolation)

(b) Using points x=4.5 and x=5.5.

$\log(5) \approx 0.6532125 + (5 - 4.5)\ 0.0871502 = 0.696788 \qquad \varepsilon_t = 0.3\ \%$

(c) Using points x=4 and x=6.

The entries of the above table can not be used for this interpolation.

(d) Using points x=4.5 , x=5.5 and x=6.

$\log(5) \approx 0.6532125 + (5-4.5)\ 0.0871502 + (5-4.5)(5-5.5)(-0.0077153) = 0.698717 \qquad \varepsilon_t = 0.04\ \%$

(e) Using all four points.

$$\log(5) \approx 0.60206 + (5 - 4)\ 0.102305 + (5 - 4)(5 - 4.5)(-0.0101032)$$
$$+ (5 - 4)(5 - 4.5)(5 - 5.5)(0.001194) = 0.6990149 \qquad \varepsilon_t = 0.006\ \%$$

# Errors of Newton's Interpolating Polynomials

$$f_n(x) = f(x_0) + (x - x_0)\, f[x_1, x_0] + (x - x_0)(x - x_1)\, f[x_2, x_1, x_0] + \ldots$$
$$+ (x - x_0)(x - x_1)\cdots(x - x_{n-1})\, f[x_n, x_{n-1}, \ldots, x_1, x_0]$$

- The structure of Newton's Interpolating Polynomials is similar to the Taylor series.
- Remainder (truncation error) for the Taylor series was $R_n = \dfrac{f^{n+1}(\xi)}{(n+1)!}(x_{i+1} - x_i)^{n+1}$
- Similarly the remainder for the $n^{th}$ order interpolating polynomial is

$$R_n = \frac{f^{n+1}(\xi)}{(n+1)!}(x - x_0)(x - x_1)\ldots(x - x_n)$$

where $\xi$ is somewhere in the interval containing the interpolated point x and other data points.

- But usually only the set of data points is given and the function f is not known.
- An alternative formulation uses a finite divided difference to approximate the $(n+1)^{th}$ derivative.

$$R_n \approx f[x, x_n, x_{n-1}, \ldots, x_0](x - x_0)(x - x_1)\ldots(x - x_n)$$

- But this includes f(x) which is not known.
- Error can be predicted if an additional data point ($x_{n+1}$) is availbale

$$R_n \approx f[x_{n+1}, x_n, x_{n-1}, \ldots, x_0](x - x_0)(x - x_1)\ldots(x - x_n)$$

which is nothing but $f_{n+1}(x) - f_n(x)$

# LAGRANGE INTERPOLATING POLYNOMIALS

The *Lagrange interpolating polynomial* is simply a reformulation of the Newton polynomial that avoids the computation of divided differences. It can be represented concisely as

$$f_n(x) = \sum_{i=0}^{n} L_i(x) f(x_i)$$

where Langrange function is:

$$L_i(x) = \prod_{\substack{j=0 \\ j \neq i}}^{n} \frac{x - x_j}{x_i - x_j}$$

and Π designates the "product of."

For example, the linear version ($n$ = 1) is

$$f_1(x) = \frac{x - x_1}{x_0 - x_1} f(x_0) + \frac{x - x_0}{x_1 - x_0} f(x_1)$$

and the second-order version is

$$f_2(x) = \frac{(x - x_1)(x - x_2)}{(x_0 - x_1)(x_0 - x_2)} f(x_0) + \frac{(x - x_0)(x - x_2)}{(x_1 - x_0)(x_1 - x_2)} f(x_1)$$
$$+ \frac{(x - x_0)(x - x_1)}{(x_2 - x_0)(x_2 - x_1)} f(x_2)$$

**Problem Statement.** Use a Lagrange interpolating polynomial of the first and second order to evaluate ln 2 on the basis of the data given in Example 18.2:

$$x_0 = 1 \qquad f(x_0) = 0$$
$$x_1 = 4 \qquad f(x_1) = 1.386294$$
$$x_2 = 6 \qquad f(x_2) = 1.791760$$

**Solution.** The first-order polynomial [Eq. (18.22)] can be used to obtain the estimate at $x = 2$,

$$f_1(2) = \frac{2-4}{1-4}0 + \frac{2-1}{4-1}1.386294 = 0.4620981$$

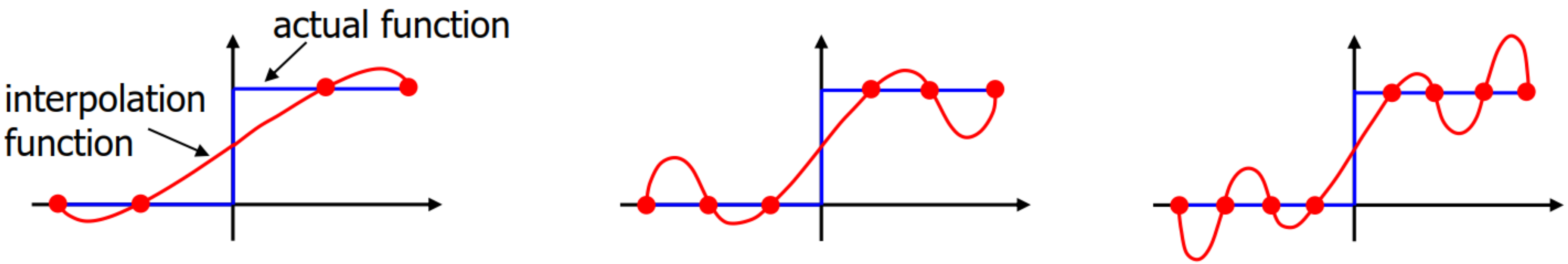
In a similar fashion, the second-order polynomial is developed as [Eq. (18.23)]

$$f_2(2) = \frac{(2-4)(2-6)}{(1-4)(1-6)}0 + \frac{(2-1)(2-6)}{(4-1)(4-6)}1.386294$$
$$+ \frac{(2-1)(2-4)}{(6-1)(6-4)}1.791760 = 0.5658444$$

As expected, both these results agree with those previously obtained using Newton's interpolating polynomial.

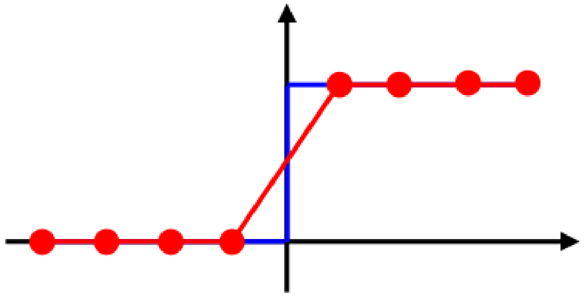
# SPLINE INTERPOLATION

- We learned how to interpolate between n+1 data points using $n^{th}$ order polynomials.
- For high number of data points (typically n > 6 or 7), high order polynomials are necessary, but sometimes they suffer from oscillatory behavior.

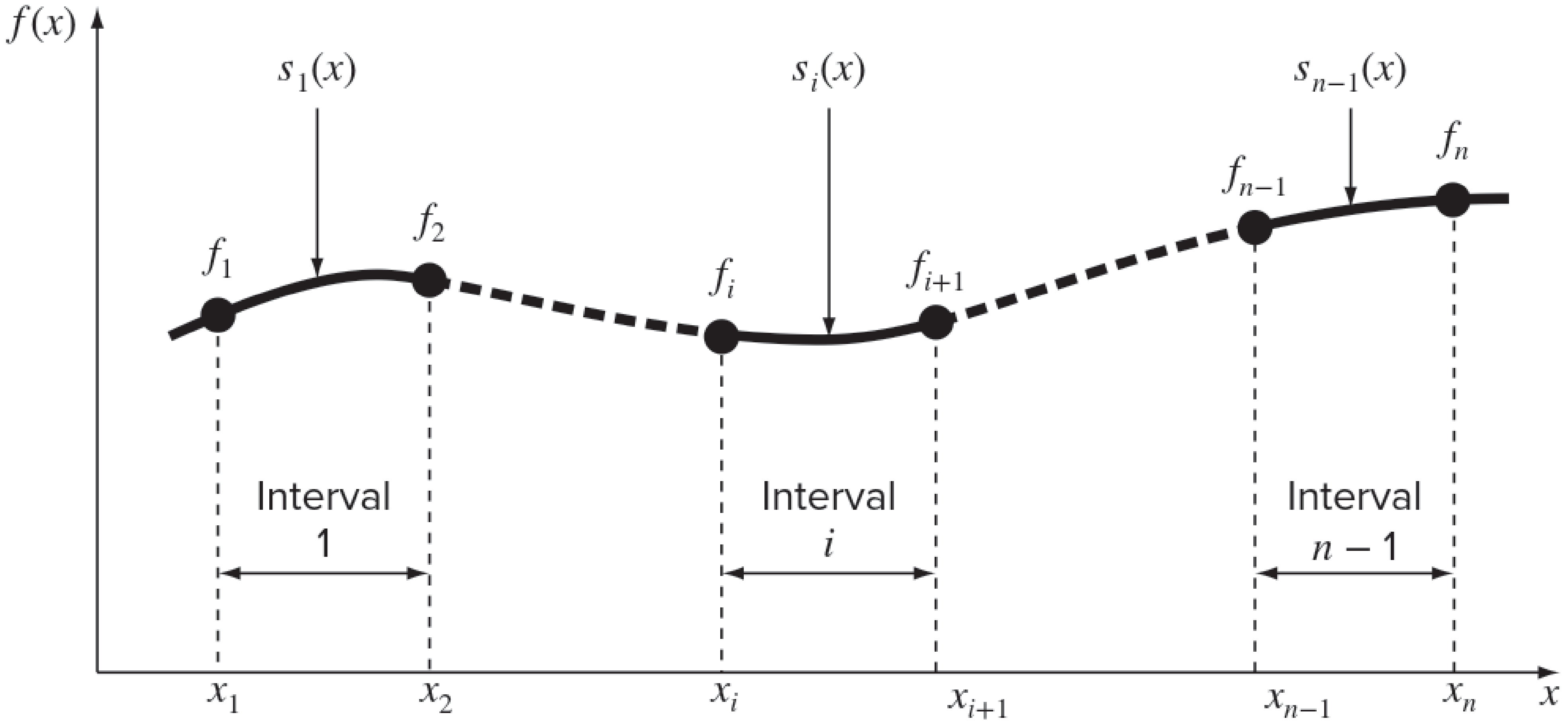



- Instead of using a single high order polynomial that passes through all data points, we can use different lower order polynomials between each data pair.
- These lower order polynomials that pass through only two points are called splines.
- Third order (cubic) splines are the most preferred ones.

first order splines :



# Linear Splines




The notation used for splines

# Linear Splines

- For $n$ data points ($i = 1, 2, \ldots, n$), there are $n - 1$ intervals. Each interval $i$ has its own spline function, $s_i(x)$.
- For linear splines, each function is merely the straight line connecting the two points at each end of the interval, which is formulated as

$$s_i(x) = a_i + b_i(x - x_i)$$

where $a_i$ is the intercept, which is defined as $a_i = f_i = f(x_i)$

and $b_i$ is the slope of the straight line connecting the points

$$b_i = \frac{f_{i+1} - f_i}{x_{i+1} - x_i}$$

$$\boxed{s_i(x) = f_i + \frac{f_{i+1} - f_i}{x_{i+1} - x_i}(x - x_i)}$$

These equations can be used to evaluate the function at any point between $x_1$ and $x_n$ by first locating the interval within which the point lies.

Problem Statement. Fit the data in Table 18.1 with first-order splines. Evaluate the function at $x = 5$.

**TABLE 18.1** Data to be fit with spline functions.

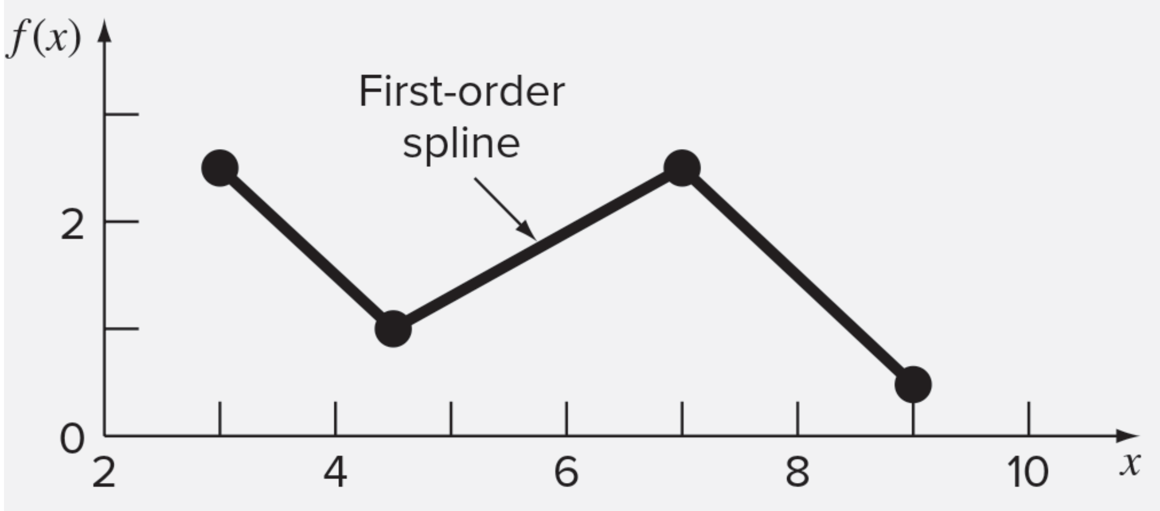
| $i$ | $x_i$ | $f_i$ |
|---|---|---|
| 1 | 3.0 | 2.5 |
| 2 | 4.5 | 1.0 |
| 3 | 7.0 | 2.5 |
| 4 | 9.0 | 0.5 |

## Solution.

For example, for the second interval from $x$ = 4.5 to $x$ = 7, the function is

$$s_2(x) = 1.0 + \frac{2.5 - 1.0}{7.0 - 4.5}(x - 4.5)$$

$$s_2(x) = 1.0 + \frac{2.5 - 1.0}{7.0 - 4.5}(5 - 4.5) = 1.3$$



# Quadratic Splines

- To ensure that the *n*th derivatives are continuous at the knots, a spline of at least $n + 1$ order must be used.
- Third-order polynomials or cubic splines that ensure continuous first and second derivatives are most frequently used in practice.
- The objective in quadratic splines is to derive a second-order polynomial for each interval between data points.

$$s_i(x) = a_i + b_i(x - x_i) + c_i(x - x_i)^2$$

For $n$ data points ($i = 1, 2, \ldots, n$), there are $n - 1$ intervals and, consequently, $3(n - 1)$ unknown constants (the *a*'s, *b*'s, and *c*'s) to evaluate. Therefore, $3(n - 1)$ equations or conditions are required to evaluate the unknowns.

1. The function must pass through all the points. This is called a *continuity condition*. It can be expressed mathematically as

$$f_i = a_i + b_i(x_i - x_i) + c_i(x_i - x_i)^2$$

which simplifies to

$$a_i = f_i \tag{18.32}$$

$$s_i(x) = f_i + b_i(x - x_i) + c_i(x - x_i)^2$$

2. The function values of adjacent polynomials must be equal at the knots.

$$f_i + b_i h_i + c_i h_i^2 = f_{i+1} \qquad\qquad h_i = x_{i+1} - x_i$$

3. The first derivatives at the interior nodes must be equal.

$$s_i'(x) = b_i + 2c_i(x - x_i) \qquad\qquad b_i + 2c_i h_i = b_{i+1}$$

4. Assume that the second derivative is zero at the first point. $\qquad c_1 = 0$

**Problem Statement.** Fit quadratic splines to the same data employed in previous example (Table 18.1). Use the results to estimate the value of the function at $x = 5$.

**TABLE 18.1** Data to be fit with spline functions.

| $i$ | $x_i$ | $f_i$ |
|---|---|---|
| 1 | 3.0 | 2.5 |
| 2 | 4.5 | 1.0 |
| 3 | 7.0 | 2.5 |
| 4 | 9.0 | 0.5 |

**Solution.** For the present problem, we have four data points and $n = 3$ intervals. Therefore, after applying the continuity condition and the zero second-derivative condition, this means that $2(4 - 1) - 1 = 5$ conditions are required. Equation (18.34) is written for $i = 1$ through 3 (with $c_1 = 0$) to give

$$f_1 + b_1 h_1 = f_2$$

$$f_2 + b_2 h_2 + c_2 h_2^2 = f_3$$

$$f_3 + b_3 h_3 + c_3 h_3^2 = f_4$$

Continuity of derivatives, Eq. (18.35), creates an additional $3 - 1 = 2$ conditions (again, recall that $c_1 = 0$):

$$b_1 = b_2$$

$$b_2 + 2c_2 h_2 = b_3$$

The necessary function and interval width values are

$$f_1 = 2.5 \qquad h_1 = 4.5 - 3.0 = 1.5$$
$$f_2 = 1.0 \qquad h_2 = 7.0 - 4.5 = 2.5$$
$$f_3 = 2.5 \qquad h_3 = 9.0 - 7.0 = 2.0$$
$$f_4 = 0.5$$

These values can be substituted into the conditions, which can be expressed in matrix form as

$$\begin{bmatrix} 1.5 & 0 & 0 & 0 & 0 \\ 0 & 2.5 & 6.25 & 0 & 0 \\ 0 & 0 & 0 & 2 & 4 \\ 1 & -1 & 0 & 0 & 0 \\ 0 & 1 & 5 & -1 & 0 \end{bmatrix} \begin{Bmatrix} b_1 \\ b_2 \\ c_2 \\ b_3 \\ c_3 \end{Bmatrix} = \begin{Bmatrix} -1.5 \\ 1.5 \\ -2 \\ 0 \\ 0 \end{Bmatrix}$$

These equations can be solved with the results:

$$b_1 = -1$$
$$b_2 = -1 \qquad c_2 = 0.64$$
$$b_3 = 2.2 \qquad c_3 = -1.6$$

These results, along with the values for the *a*'s [Eq. (18.32)], can be substituted into the original quadratic equations to develop the following quadratic splines for each interval:

$$s_1(x) = 2.5 - (x - 3)$$

$$s_2(x) = 1.0 - (x - 4.5) + 0.64(x - 4.5)^2$$

$$s_3(x) = 2.5 + 2.2(x - 7.0) - 1.6(x - 7.0)^2$$

Because $x = 5$ lies in the second interval, we use $s_2$ to make the prediction,

$$s_2(5) = 1.0 - (5 - 4.5) + 0.64(5 - 4.5)^2 = 0.66$$

# Cubic Splines

- Cubic splines are most frequently used in practice.
- Cubic splines are preferred because they provide the simplest representation that exhibits the desired appearance of smoothness.
- The objective with cubic splines is to derive a third-order polynomial for each interval between knots as represented generally by

$$s_i(x) = a_i + b_i(x - x_i) + c_i(x - x_i)^2 + d_i(x - x_i)^3$$

Thus, for n data points (i = 1, 2, . . . , n), there are n − 1 intervals and 4(n − 1) unknown coefficients to evaluate.

Consequently, 4(n − 1) conditions are required for their evaluation.

The final equations can now be written in matrix form as

$$\begin{bmatrix} 1 & & & & & \\ h_1 & 2(h_1+h_2) & h_2 & & & \\ & \ddots & \ddots & \ddots & & \\ & & h_{n-2} & 2(h_{n-2}+h_{n-1}) & h_{n-1} \\ & & & & 1 \end{bmatrix} \begin{Bmatrix} c_1 \\ c_2 \\ \vdots \\ c_{n-1} \\ c_n \end{Bmatrix} = \begin{Bmatrix} 0 \\ 3(f[x_3, x_2] - f[x_2, x_1]) \\ \vdots \\ 3(f[x_n, x_{n-1}] - f[x_{n-1}, x_{n-2}]) \\ 0 \end{Bmatrix}$$

$$h_i = x_{i+1} - x_i$$

$$f[x_i, x_j] = \frac{f_i - f_j}{x_i - x_j}$$

$$a_i = f_i$$

$$b_i = \frac{f_{i+1} - f_i}{h_i} - \frac{h_i}{3}(2c_i + c_{i+1})$$

$$d_i = \frac{c_{i+1} - c_i}{3h_i}$$

**Problem Statement.** Fit cubic splines to the same data used in previous examples (Table 18.1). Utilize the results to estimate the value of the function at $x = 5$.

**TABLE 18.1** Data to be fit with spline functions.

| $i$ | $x_i$ | $f_i$ |
|---|---|---|
| 1 | 3.0 | 2.5 |
| 2 | 4.5 | 1.0 |
| 3 | 7.0 | 2.5 |
| 4 | 9.0 | 0.5 |

**Solution.** The first step is to generate the set of simultaneous equations that will be utilized to determine the $c$ coefficients:

$$\begin{bmatrix} 1 & & & \\ h_1 & 2(h_1 + h_2) & h_2 & \\ & h_2 & 2(h_2 + h_3) & h_3 \\ & & & 1 \end{bmatrix} \begin{Bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 3(f[x_3, x_2] - f[x_2, x_1]) \\ 3(f[x_4, x_3] - f[x_3, x_2]) \\ 0 \end{Bmatrix}$$

The necessary function and interval width values are

$$\begin{aligned} f_1 &= 2.5 & h_1 &= 4.5 - 3.0 = 1.5 \\ f_2 &= 1.0 & h_2 &= 7.0 - 4.5 = 2.5 \\ f_3 &= 2.5 & h_3 &= 9.0 - 7.0 = 2.0 \\ f_4 &= 0.5 & & \end{aligned}$$

$$
\begin{bmatrix} 1 & & & \\ 1.5 & 8 & 2.5 & \\ & 2.5 & 9 & 2 \\ & & & 1 \end{bmatrix} \begin{Bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 4.8 \\ -4.8 \\ 0 \end{Bmatrix}
$$

$c_1 = 0$ $\quad$ $c_2 = 0.839543726$

$c_3 = -0.766539924$ $\quad$ $c_4 = 0$

<u>Compute the *b*'s and *d*'s:</u>

$b_1 = -1.419771863$ $\quad$ $d_1 = 0.186565272$

$b_2 = -0.160456274$ $\quad$ $d_2 = -0.214144487$

$b_3 = 0.022053232$ $\quad$ $d_3 = 0.127756654$

$$s_1(x) = 2.5 - 1.419771863(x - 3) + 0.186565272(x - 3)^3$$
$$s_2(x) = 1.0 - 0.160456274(x - 4.5) + 0.839543726(x - 4.5)^2$$
$$\quad - 0.214144487(x - 4.5)^3$$
$$s_3(x) = 2.5 + 0.022053232(x - 7.0) - 0.766539924(x - 7.0)^2$$
$$\quad + 0.127756654(x - 7.0)^3$$

The three equations can then be employed to compute values within each interval. For example, the value at $x = 5$, which falls within the second interval, is calculated as

$$s_2(5) = 1.0 - 0.160456274(5 - 4.5) + 0.839543726(5 - 4.5)^2 - 0.214144487(5 - 4.5)^3$$
$$= 1.102889734.$$