

CHAPTER 2

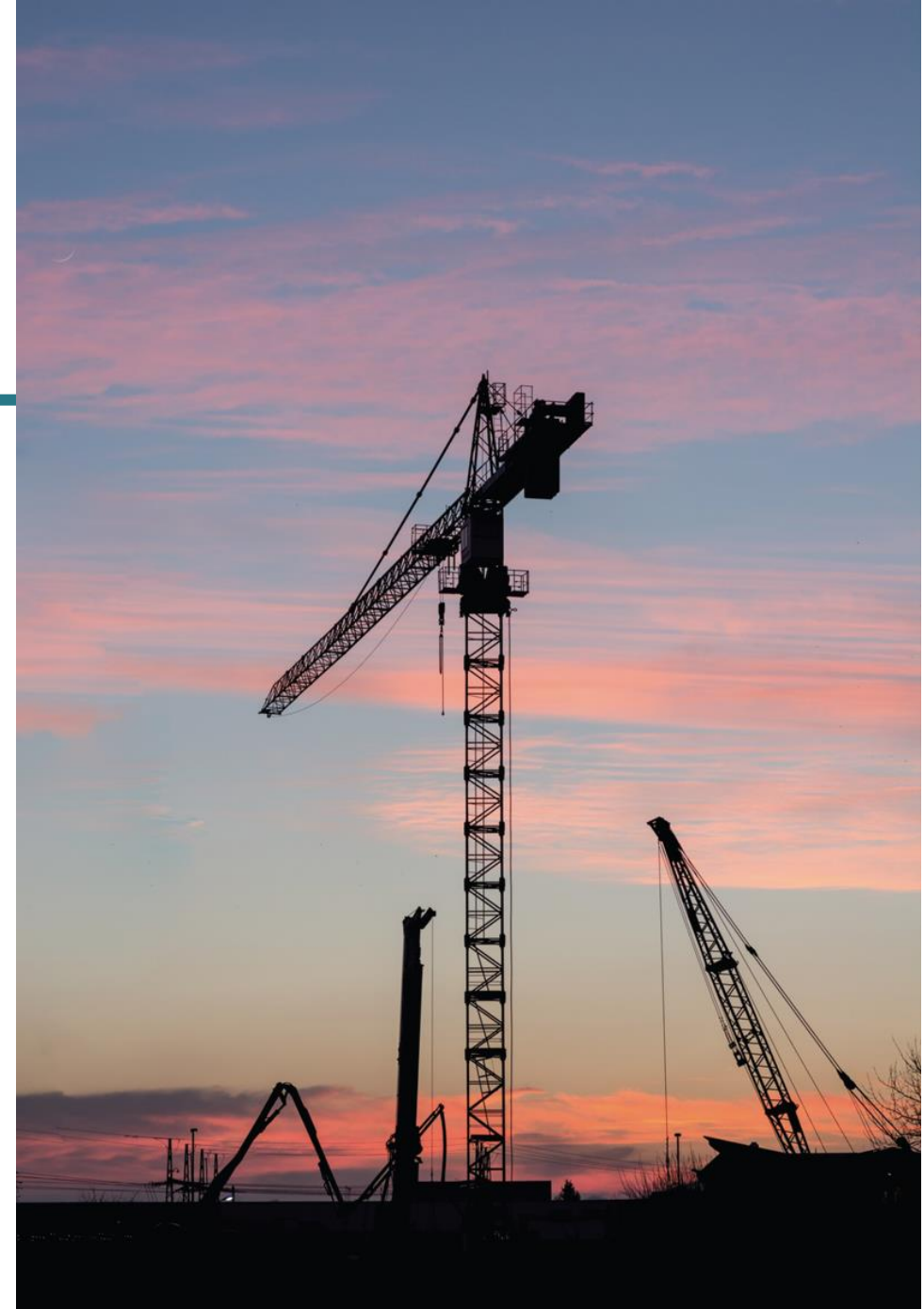
FORCE SYSTEMS

SECTION B Three-Dimensional Force Systems

2/7 Rectangular Components

2/8 Moment and Couple

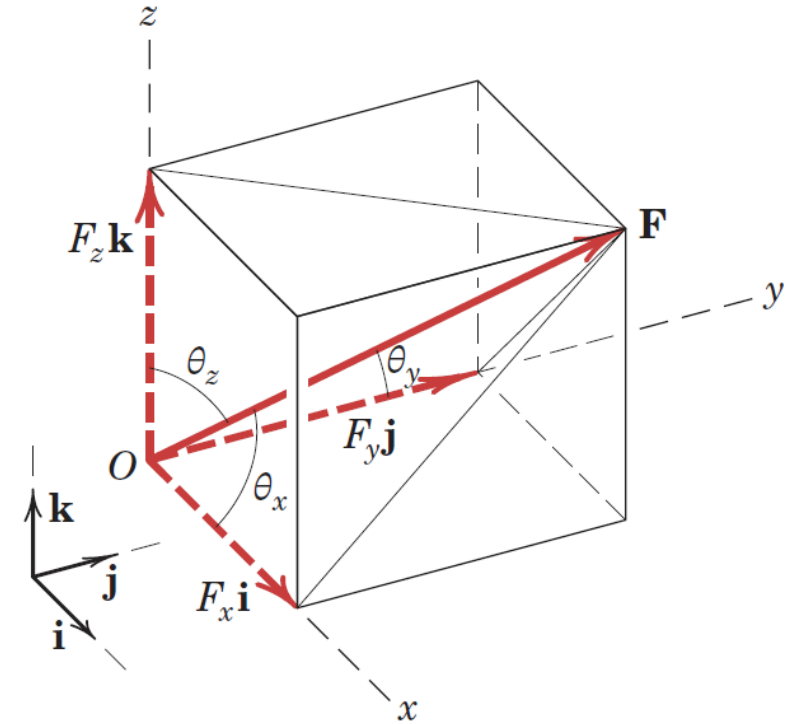
2/9 Resultants



Article 2/7 Rectangular Components (3D)

- Illustration and Equations of Interest

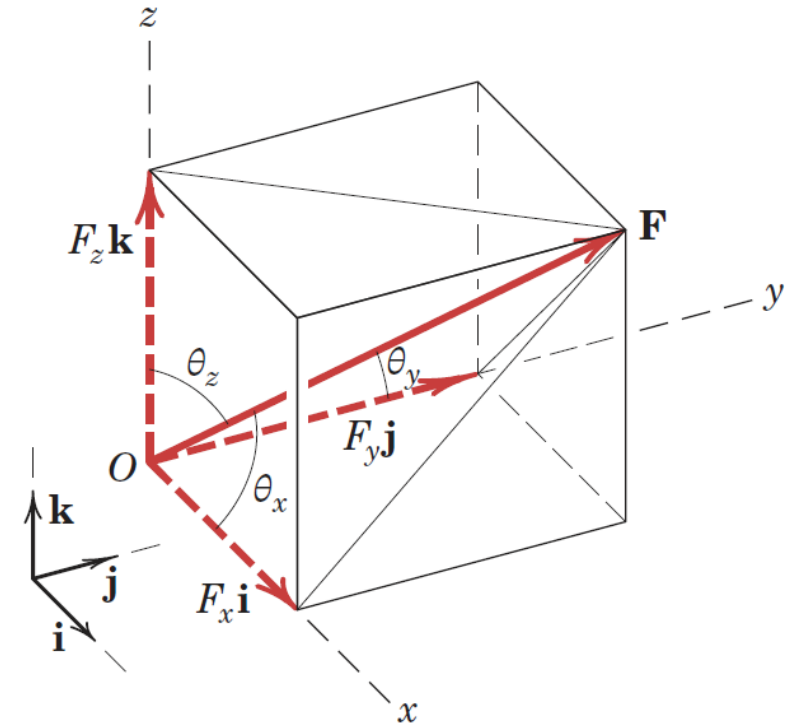
$$\begin{array}{ll} F_x = F \cos \theta_x & F = \sqrt{F_x^2 + F_y^2 + F_z^2} \\ F_y = F \cos \theta_y & \mathbf{F} = F_x \mathbf{i} + F_y \mathbf{j} + F_z \mathbf{k} \\ F_z = F \cos \theta_z & \mathbf{F} = F(\mathbf{i} \cos \theta_x + \mathbf{j} \cos \theta_y + \mathbf{k} \cos \theta_z) \end{array}$$



Article 2/7 – Rectangular Components (cont.)

- Magnitude and Direction Format
 - $\mathbf{F} = F\mathbf{n}_f$ where $\mathbf{n}_f$ is a unit vector in the direction of $\mathbf{F}$.
 - $\mathbf{n}_f = \cos \theta_x \mathbf{i} + \cos \theta_y \mathbf{j} + \cos \theta_z \mathbf{k}$

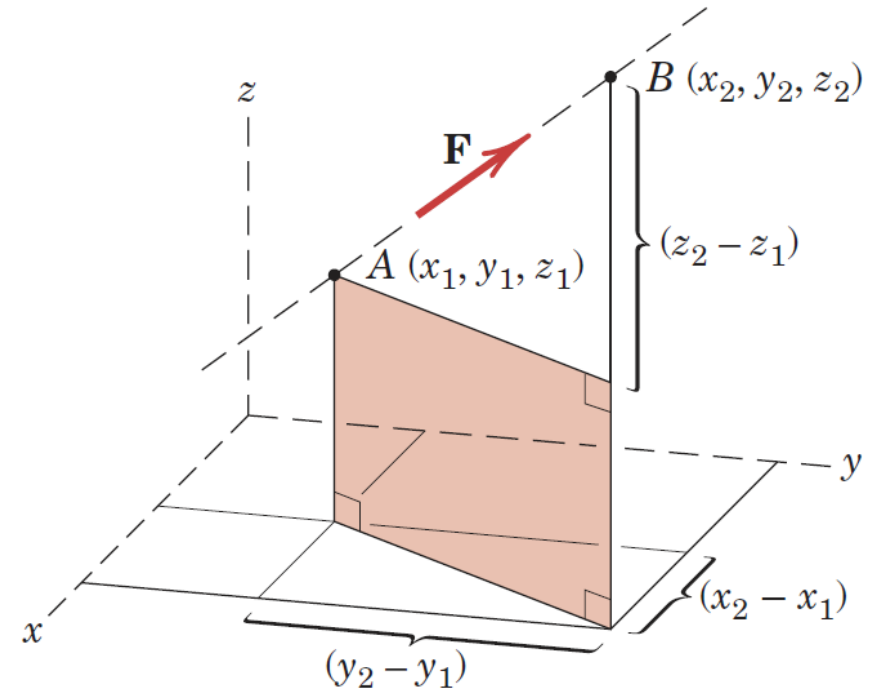
- Direction Cosine Format
 - $\mathbf{F} = F\mathbf{n}_f = F(l\mathbf{i} + m\mathbf{j} + n\mathbf{k})$
 - $l = \cos \theta_x$
 - $m = \cos \theta_y$
 - $n = \cos \theta_z$
 - $l^2 + m^2 + n^2 = 1$



Article 2/7 – Writing Vector Components (1 of 2)

- Specification by two points on the line of action of the force.

$$\mathbf{F} = F\mathbf{n}_F = F \frac{\vec{AB}}{AB} = F \frac{(x_2 - x_1)\mathbf{i} + (y_2 - y_1)\mathbf{j} + (z_2 - z_1)\mathbf{k}}{\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}}$$



Article 2/7 – Writing Vector Components (2 of 2)

- Specification by two angles which orient the line of action of the force.

- Horizontal and Vertical Components

$$F_{xy} = F \cos \phi$$

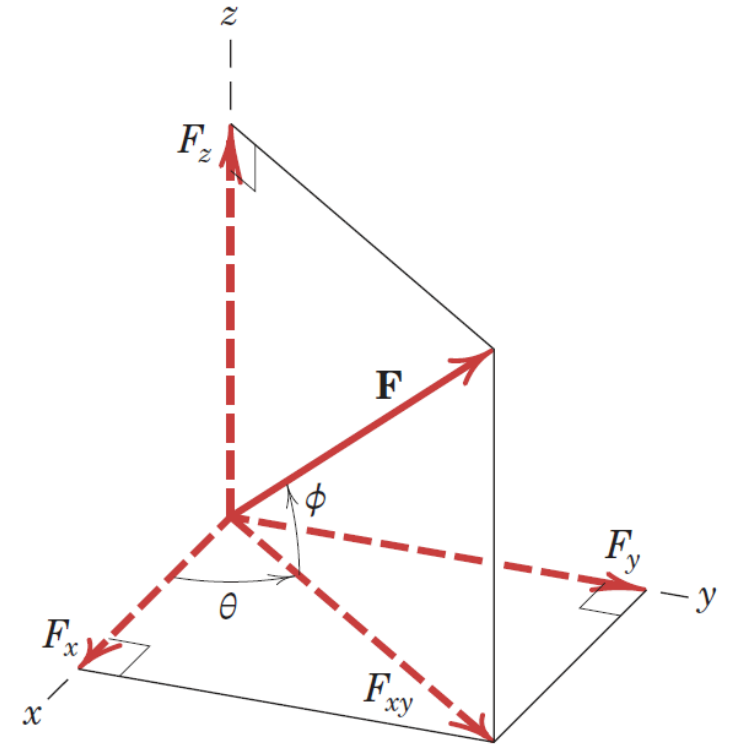
$$F_z = F \sin \phi$$

- x- and y-Components

$$F_x = F_{xy} \cos \theta = F \cos \phi \cos \theta$$

$$F_y = F_{xy} \sin \theta = F \cos \phi \sin \theta$$

- Other Combinations of Angles



Article 2/7 – The Dot Product

- Definitions and Illustration
 - $\mathbf{P} \cdot \mathbf{Q} = \mathbf{Q} \cdot \mathbf{P} = PQ \cos \theta$
 - $\theta = \cos^{-1}(\mathbf{P} \cdot \mathbf{Q} / PQ)$
- Mathematical Features of the Dot Product

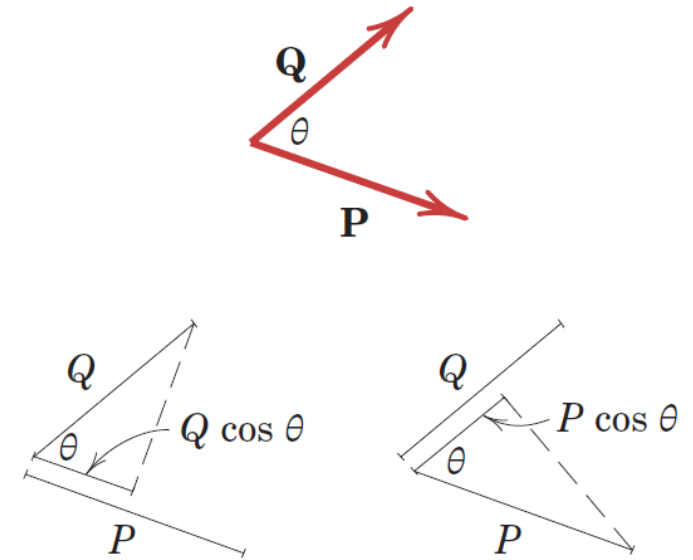
$$\mathbf{i} \cdot \mathbf{i} = \mathbf{j} \cdot \mathbf{j} = \mathbf{k} \cdot \mathbf{k} = 1$$

$$\mathbf{i} \cdot \mathbf{j} = \mathbf{j} \cdot \mathbf{i} = \mathbf{i} \cdot \mathbf{k} = \mathbf{k} \cdot \mathbf{i} = \mathbf{j} \cdot \mathbf{k} = \mathbf{k} \cdot \mathbf{j} = 0$$

$$\mathbf{P} \cdot \mathbf{Q} = (P_x \mathbf{i} + P_y \mathbf{j} + P_z \mathbf{k}) \cdot (Q_x \mathbf{i} + Q_y \mathbf{j} + Q_z \mathbf{k})$$

$$= P_x Q_x + P_y Q_y + P_z Q_z$$

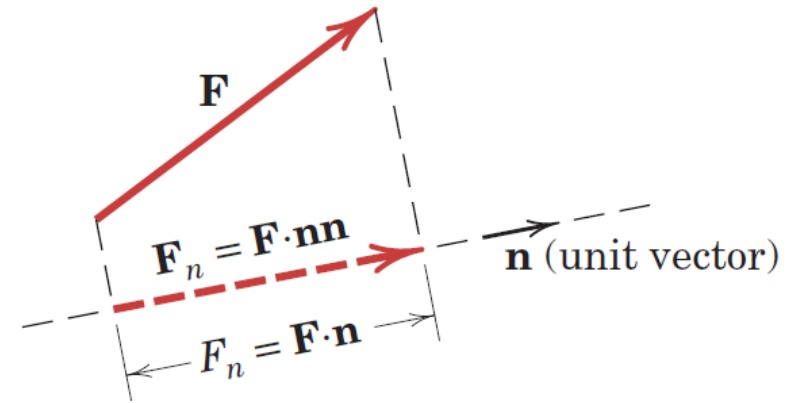
$$\mathbf{P} \cdot \mathbf{P} = P_x^2 + P_y^2 + P_z^2$$



Article 2/7 – Finding Projections of Forces onto Lines

- Scalar Projection of a Force onto a line, F_n

1. Write the force as a vector.
2. Write a unit vector in the direction of the line.
3. Dot the force vector with the unit vector.



- Vector Projection of a Force, $\mathbf{F}_n$

1. Write the scalar projection of the force onto the line.
2. Multiply the scalar projection by the unit vector for the line.

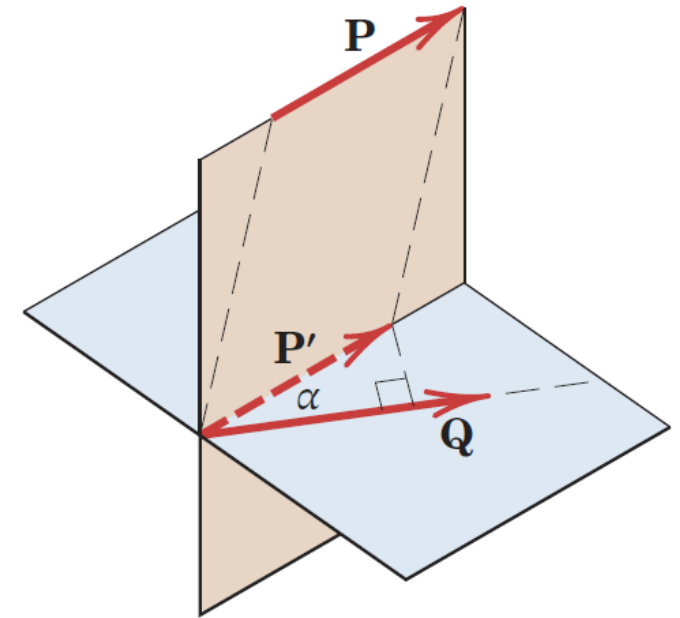
Article 2/7 – Finding the Angle between Two Vectors

- Dot Product Reminder

- $\mathbf{P} \cdot \mathbf{Q} = \mathbf{Q} \cdot \mathbf{P} = PQ \cos \alpha$
- $\alpha = \cos^{-1}(\mathbf{P} \cdot \mathbf{Q} / PQ)$

- Solution Steps

1. Write each vector.
2. Take a dot product between the vectors.
3. Divide the dot product by the product of the magnitudes of the vectors.
4. Take the inverse cosine of this ratio.

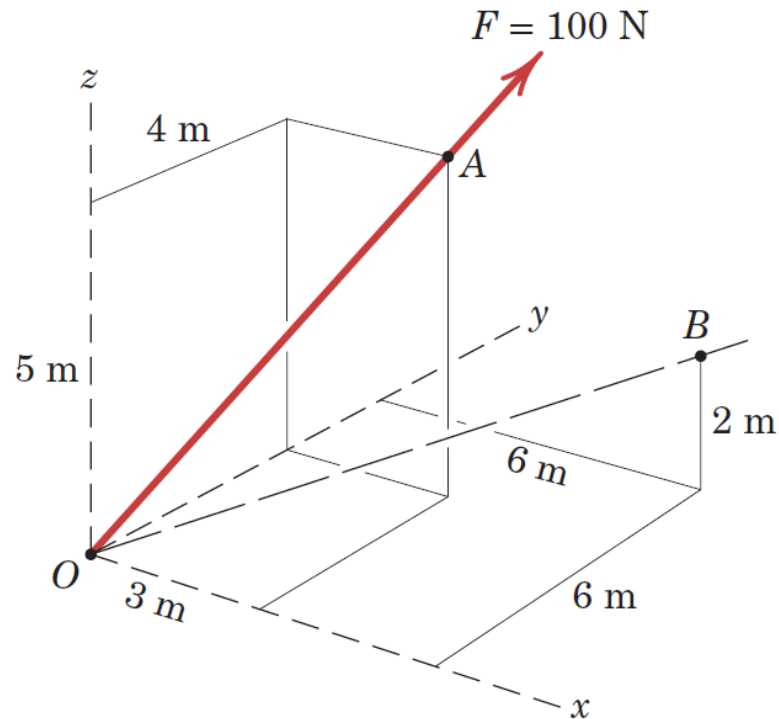


- This process is more easily carried out with unit vectors.

Article 2/7 – Sample Problem 2/10 (1 of 4)

- Problem Statement

A force $\mathbf{F}$ with a magnitude of 100 N is applied at the origin O of the axes x - y - z as shown. The line of action of $\mathbf{F}$ passes through a point A whose coordinates are 3 m, 4 m, and 5 m. Determine (a) the x , y , and z scalar components of $\mathbf{F}$, (b) the projection F_{xy} of $\mathbf{F}$ on the x - y plane, and (c) the projection F_{OB} of $\mathbf{F}$ along the line OB .



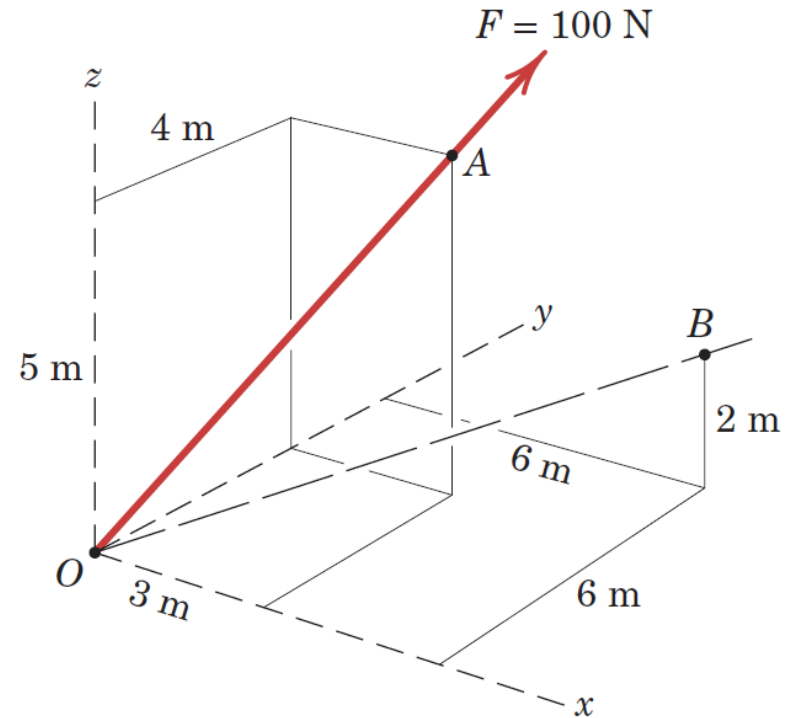
Article 2/7 – Sample Problem 2/10 (2 of 4)

- Scalar Components of F

$$\begin{aligned}\mathbf{F} &= F\mathbf{n}_{OA} = F \frac{\vec{OA}}{OA} = 100 \left[\frac{3\mathbf{i} + 4\mathbf{j} + 5\mathbf{k}}{\sqrt{3^2 + 4^2 + 5^2}} \right] \\ &= 100[0.424\mathbf{i} + 0.566\mathbf{j} + 0.707\mathbf{k}] \\ &= 42.4\mathbf{i} + 56.6\mathbf{j} + 70.7\mathbf{k} \text{ N}\end{aligned}$$

The desired scalar components are thus

$$F_x = 42.4 \text{ N} \quad F_y = 56.6 \text{ N} \quad F_z = 70.7 \text{ N} \quad \textcircled{1} \quad \textit{Ans.}$$



① In this example all scalar components are positive. Be prepared for the case where a direction cosine, and hence the scalar component, is negative.

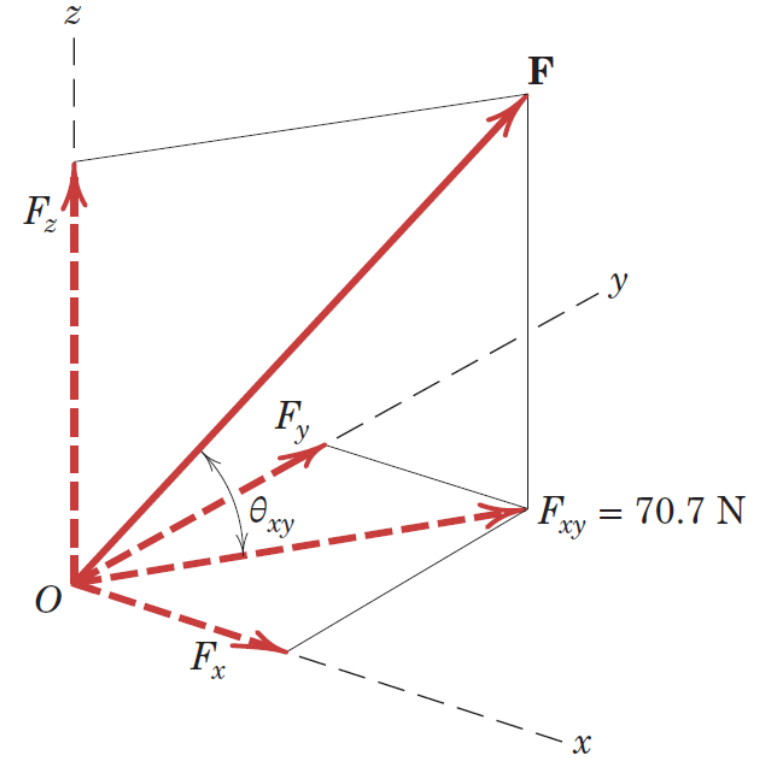
Article 2/7 – Sample Problem 2/10 (3 of 4)

- Projection of F into the x - y Plane

$$\cos \theta_{xy} = \frac{\sqrt{3^2 + 4^2}}{\sqrt{3^2 + 4^2 + 5^2}} = 0.707$$

so that $F_{xy} = F \cos \theta_{xy} = 100(0.707) = 70.7 \text{ N}$

Ans.



Article 2/7 – Sample Problem 2/10 (4 of 4)

- Projection of F onto Line OB

$$\mathbf{n}_{OB} = \frac{\vec{OB}}{OB} = \frac{6\mathbf{i} + 6\mathbf{j} + 2\mathbf{k}}{\sqrt{6^2 + 6^2 + 2^2}} = 0.688\mathbf{i} + 0.688\mathbf{j} + 0.229\mathbf{k}$$

The scalar projection of $\mathbf{F}$ on OB is

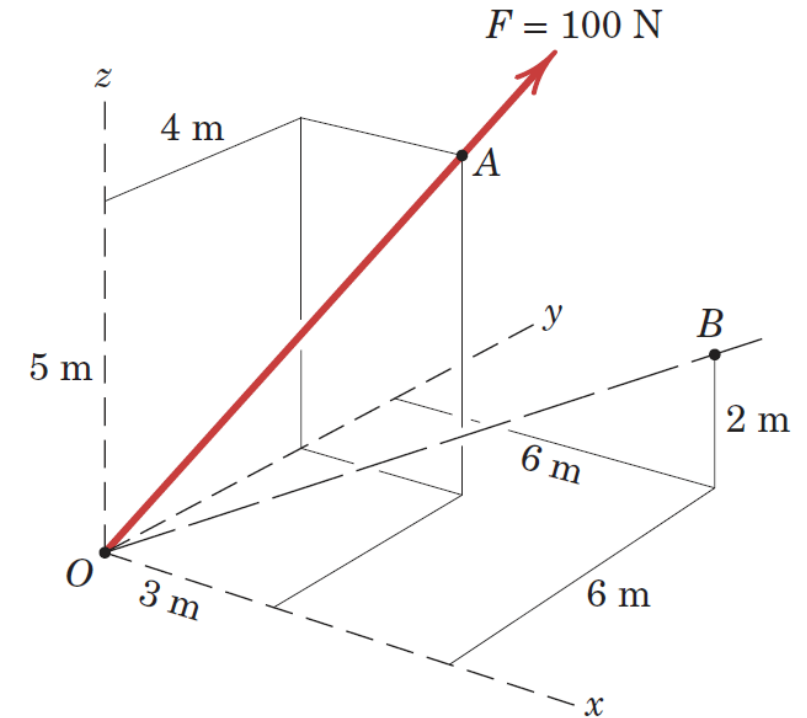
$$\begin{aligned} F_{OB} &= \mathbf{F} \cdot \mathbf{n}_{OB} = (42.4\mathbf{i} + 56.6\mathbf{j} + 70.7\mathbf{k}) \cdot (0.688\mathbf{i} + 0.688\mathbf{j} + 0.229\mathbf{k}) \quad \textcircled{2} \\ &= (42.4)(0.688) + (56.6)(0.688) + (70.7)(0.229) \\ &= 84.4 \text{ N} \end{aligned}$$

Ans.

If we wish to express the projection as a vector, we write

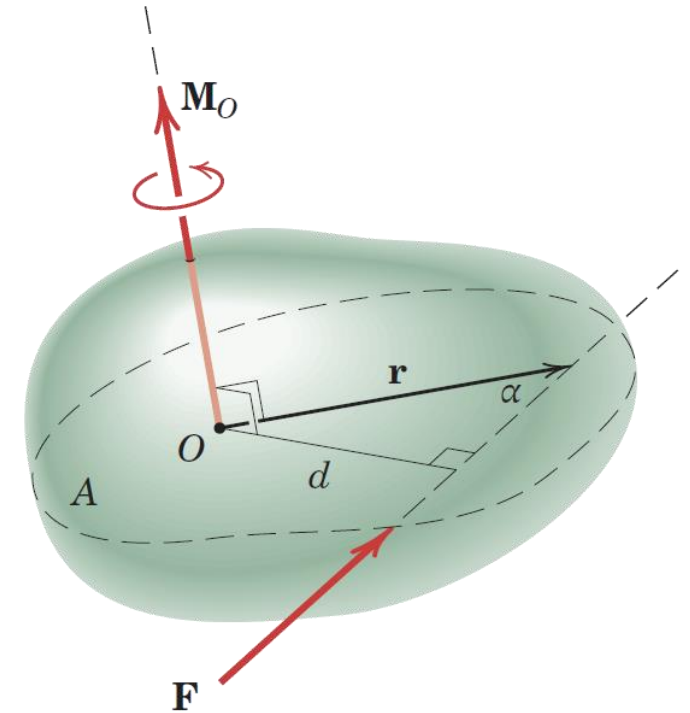
$$\begin{aligned} \mathbf{F}_{OB} &= \mathbf{F} \cdot \mathbf{n}_{OB} \mathbf{n}_{OB} \\ &= 84.4(0.688\mathbf{i} + 0.688\mathbf{j} + 0.229\mathbf{k}) \\ &= 58.1\mathbf{i} + 58.1\mathbf{j} + 19.35\mathbf{k} \text{ N} \end{aligned}$$

② The dot product automatically finds the projection or scalar component of $\mathbf{F}$ along line OB as shown.



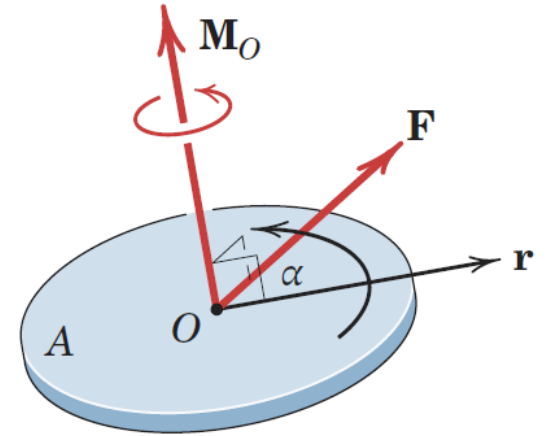
Article 2/8 Moment and Couple (3D)

- Moments in Three Dimensions
 - Operate Identically to Moments in Two Dimensions
 - More Complicated to Visualize
- Scalar Approach: $M_O = Fd$
 - More Difficult to Accomplish
 - Lacks Sign Information
- Vector Approach: $\mathbf{M}_O = \mathbf{r} \times \mathbf{F}$
 - Easy to Compute
 - Sign Information is Included Automatically



Article 2/8 – Right-Hand Rule Reminder

- Direction and Sense of the Moment
 - Established by Right-Hand Rule
 - Perpendicular to the Plane which Contains $\mathbf{r}$ and $\mathbf{F}$
 - Cross Product Order is Essential



Article 2/8 – Cross Products (1 of 2)

- Definitions and Illustration

$$\begin{aligned}\mathbf{P} \times \mathbf{Q} &= (P_x \mathbf{i} + P_y \mathbf{j} + P_z \mathbf{k}) \times (Q_x \mathbf{i} + Q_y \mathbf{j} + Q_z \mathbf{k}) \\ &= (P_y Q_z - P_z Q_y) \mathbf{i} + (P_z Q_x - P_x Q_z) \mathbf{j} + (P_x Q_y - P_y Q_x) \mathbf{k}\end{aligned}$$

$$|\mathbf{P} \times \mathbf{Q}| = PQ \sin \theta$$

- Mathematical Features of the Cross Product

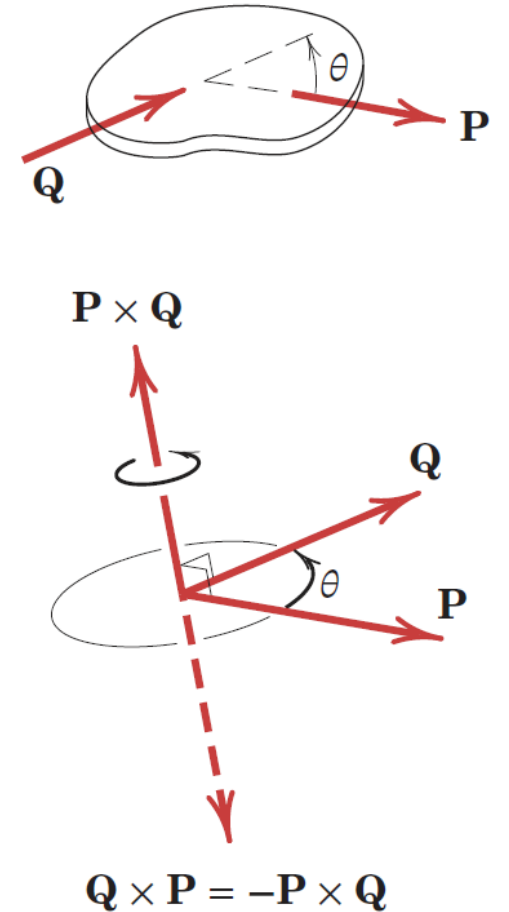
Distributive law $\mathbf{P} \times (\mathbf{Q} + \mathbf{R}) = \mathbf{P} \times \mathbf{Q} + \mathbf{P} \times \mathbf{R}$

From the definition of the cross product, using a *right-handed coordinate system*, we get

$$\mathbf{i} \times \mathbf{j} = \mathbf{k} \quad \mathbf{j} \times \mathbf{k} = \mathbf{i} \quad \mathbf{k} \times \mathbf{i} = \mathbf{j}$$

$$\mathbf{j} \times \mathbf{i} = -\mathbf{k} \quad \mathbf{k} \times \mathbf{j} = -\mathbf{i} \quad \mathbf{i} \times \mathbf{k} = -\mathbf{j}$$

$$\mathbf{i} \times \mathbf{i} = \mathbf{j} \times \mathbf{j} = \mathbf{k} \times \mathbf{k} = \mathbf{0}$$



Article 2/8 – Cross Products (2 of 2)

- Calculation via Determinant

$$\mathbf{P} \times \mathbf{Q} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ P_x & P_y & P_z \\ Q_x & Q_y & Q_z \end{vmatrix}$$

$$\begin{aligned} \mathbf{P} \times \mathbf{Q} &= (P_x \mathbf{i} + P_y \mathbf{j} + P_z \mathbf{k}) \times (Q_x \mathbf{i} + Q_y \mathbf{j} + Q_z \mathbf{k}) \\ &= (P_y Q_z - P_z Q_y) \mathbf{i} + (P_z Q_x - P_x Q_z) \mathbf{j} + (P_x Q_y - P_y Q_x) \mathbf{k} \end{aligned}$$

Article 2/8 – Moment made by a General Force

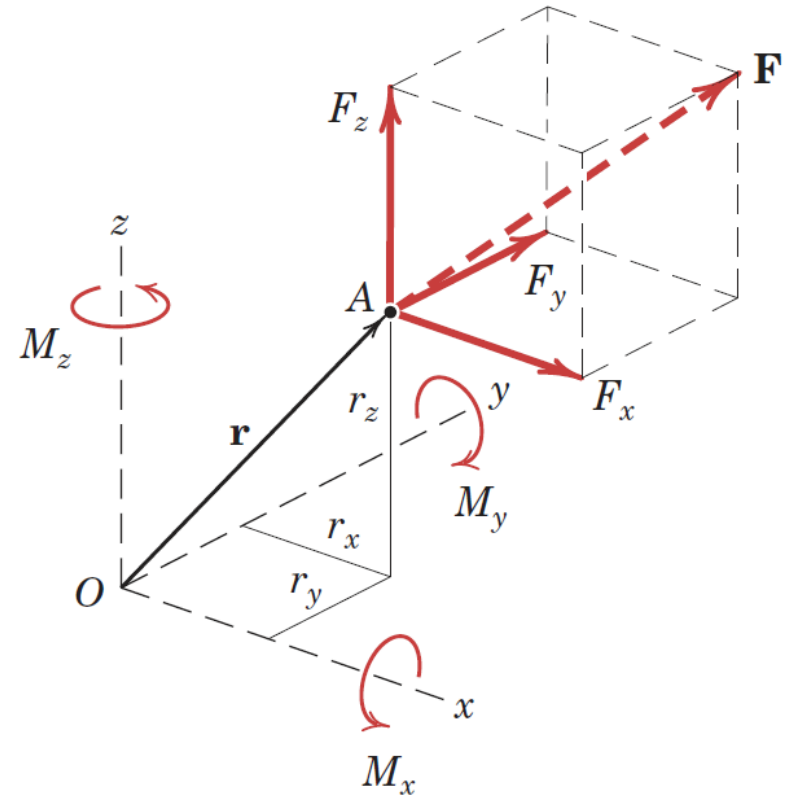
- Vector Components of $\mathbf{M}_O = \mathbf{r} \times \mathbf{F}$

$$\mathbf{M}_O = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ r_x & r_y & r_z \\ F_x & F_y & F_z \end{vmatrix}$$

$$\mathbf{M}_O = (r_y F_z - r_z F_y)\mathbf{i} + (r_z F_x - r_x F_z)\mathbf{j} + (r_x F_y - r_y F_x)\mathbf{k}$$

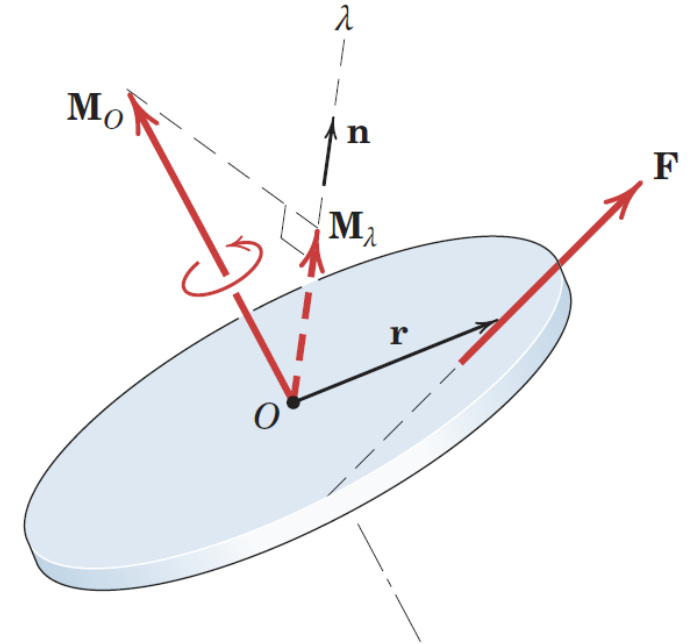
- Scalar Components

$$M_x = r_y F_z - r_z F_y \quad M_y = r_z F_x - r_x F_z \quad M_z = r_x F_y - r_y F_x$$



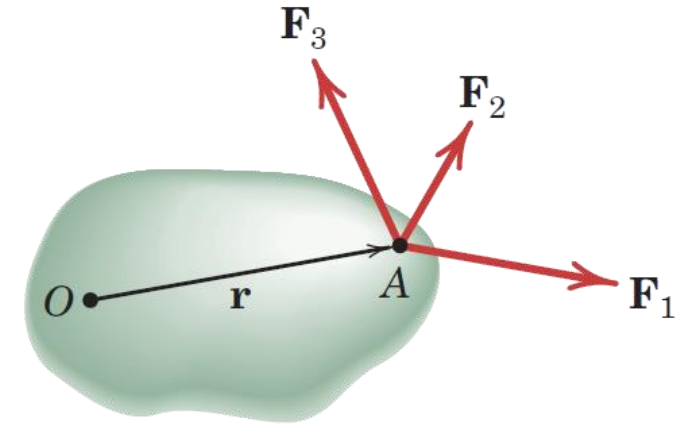
Article 2/8 – Moment about an Arbitrary Axis

- Illustration
- Scalar Expression of the Moment about an Axis, M_λ
 1. Write the force as a vector.
 2. Write a position vector from any point on the axis to any point on the line of action of the force.
 3. Compute the moment of the force about the point.
 4. Write a unit vector in the direction of the axis.
- Vector Expression of the Moment about an Axis, $\mathbf{M}_\lambda$
 1. Write the scalar expression of the moment about the axis.
 2. Multiply the scalar expression of the moment about the axis by the unit vector in the direction of the axis.



Article 2/8 – Varignon's Theorem in Three Dimensions

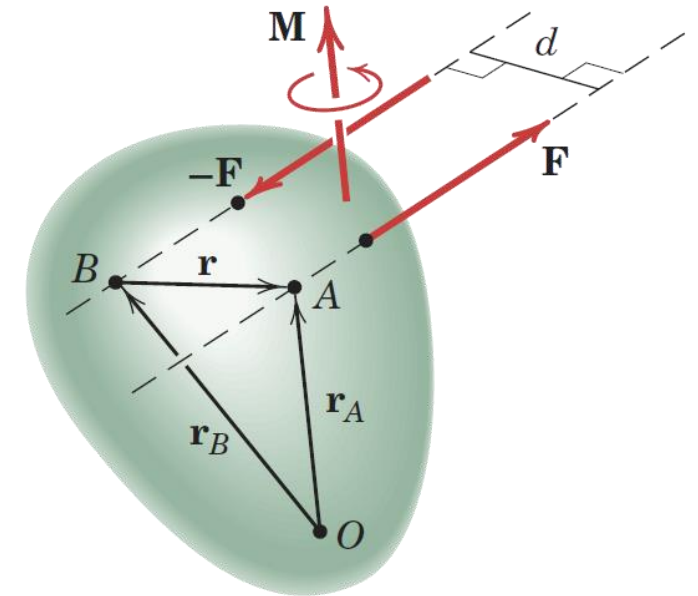
- Illustration
- Mathematics



$$\mathbf{M}_O = \Sigma(\mathbf{r} \times \mathbf{F}) = \mathbf{r} \times \mathbf{R}$$

Article 2/8 – Couples in Three Dimensions (1 of 3)

- Illustration



- Mathematics

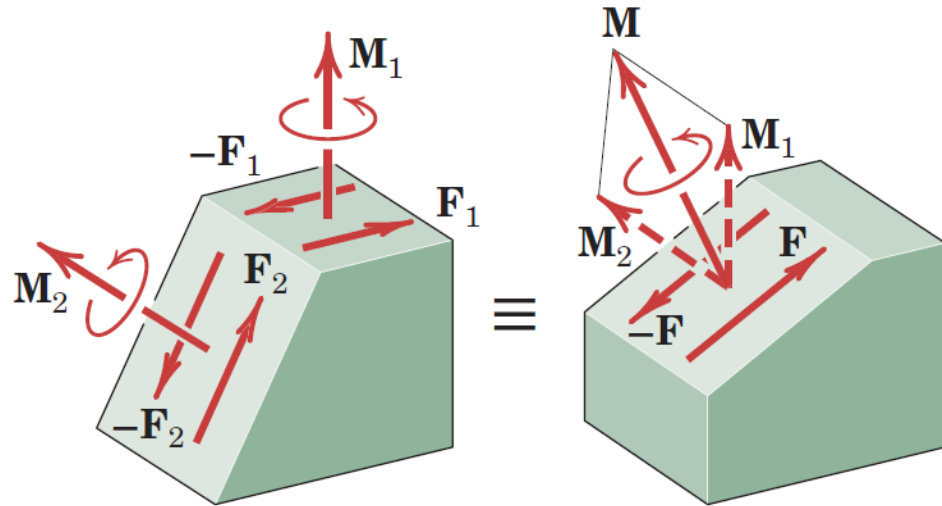
$$\mathbf{M}_O = \mathbf{r}_A \times \mathbf{F} + \mathbf{r}_B \times (-\mathbf{F}) = (\mathbf{r}_A - \mathbf{r}_B) \times \mathbf{F} = \mathbf{r} \times \mathbf{F}$$

- Comments about Couples

- Couples are free vectors.
- You can simply compute the moment of one of the forces about any point on the line of action of the other force.
- Couple vectors obey all of the corresponding mathematical rules which govern vector quantities.

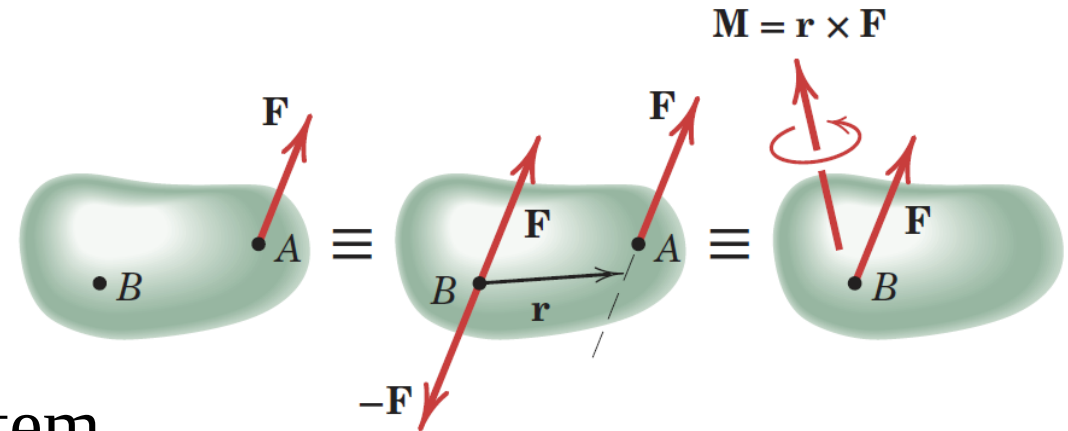
Article 2/8 – Couples in Three Dimensions (2 of 3)

- Adding Couples
 - Couples add with the Parallelogram Rule of Vector Addition



Article 2/8 – Force-Couple Systems

- Illustration of the Process



- Steps to Create a Force-Couple System

1. Write the force as a vector.
2. Compute the moment or couple which the force creates about the point.
3. Redraw the force acting at the new location.
4. Sketch the couple acting at the new location.

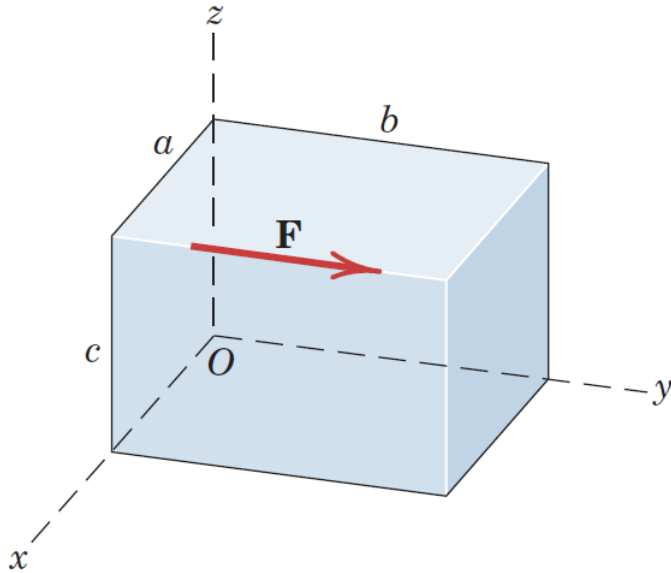
- Important Reminder

- As with two-dimensional force-couple systems, the force-couple system has the same effect on the body which the original force had. It is simply a different way to visualize the effect of the force acting at a new location.

Article 2/8 – Sample Problem 2/11 (1 of 2)

- Problem Statement

Determine the moment of force $\mathbf{F}$ about point O (a) by inspection and (b) by the formal cross-product definition $\mathbf{M}_O = \mathbf{r} \times \mathbf{F}$.



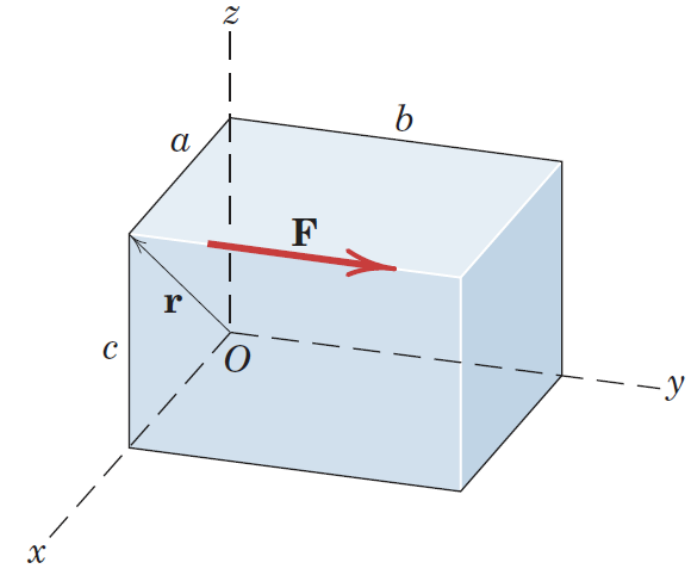
Article 2/8 – Sample Problem 2/11 (2 of 2)

- Solution by Inspection

$$\mathbf{M}_O = -cF\mathbf{i} + aF\mathbf{k} = F(-c\mathbf{i} + a\mathbf{k}) \quad \text{Ans.}$$

- Solution by Cross Product

$$\begin{aligned} \mathbf{M}_O &= \mathbf{r} \times \mathbf{F} = (a\mathbf{i} + c\mathbf{k}) \times F\mathbf{j} = aF\mathbf{k} - cF\mathbf{i} \quad \textcircled{1} \\ &= F(-c\mathbf{i} + a\mathbf{k}) \quad \text{Ans.} \end{aligned}$$

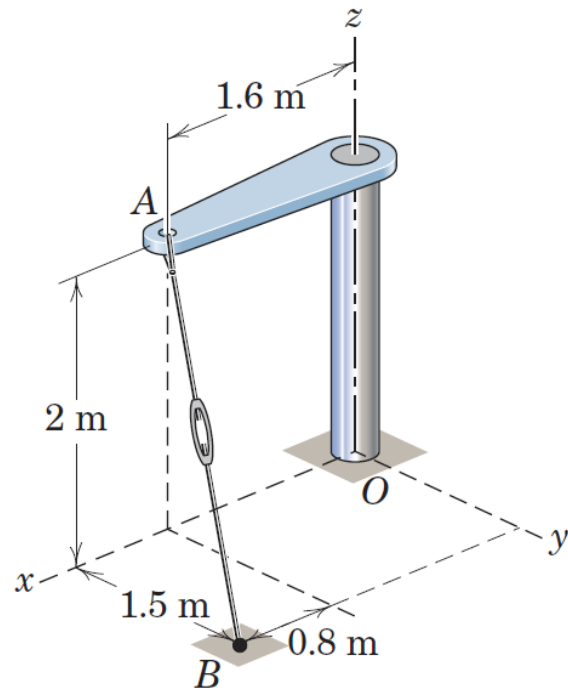


① Again we stress that $\mathbf{r}$ runs *from* the moment center *to* the line of action of $\mathbf{F}$. Another permissible, but less convenient, position vector is $\mathbf{r} = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$.

Article 2/8 – Sample Problem 2/12 (1 of 2)

- Problem Statement

The turnbuckle is tightened until the tension in cable AB is 2.4 kN. Determine the moment about point O of the cable force acting on point A and the magnitude of this moment.



Article 2/8 – Sample Problem 2/12 (2 of 2)

- Tension Vector

$$\begin{aligned}\mathbf{T} &= T\mathbf{n}_{AB} = 2.4 \left[\frac{0.8\mathbf{i} + 1.5\mathbf{j} - 2\mathbf{k}}{\sqrt{0.8^2 + 1.5^2 + 2^2}} \right] \\ &= 0.731\mathbf{i} + 1.371\mathbf{j} - 1.829\mathbf{k} \text{ kN}\end{aligned}$$

- Moment about Point O

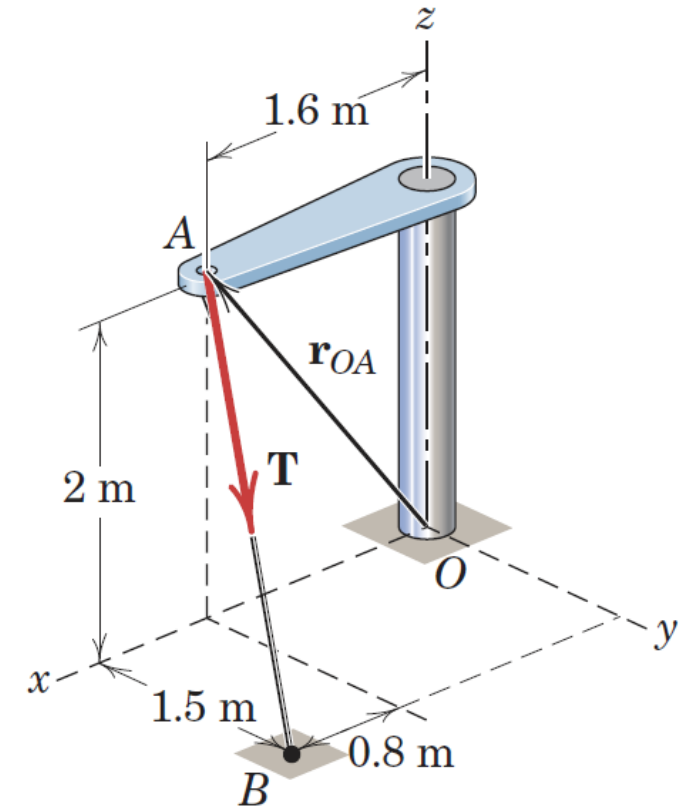
$$\begin{aligned}\mathbf{M}_O &= \mathbf{r}_{OA} \times \mathbf{T} = (1.6\mathbf{i} + 2\mathbf{k}) \times (0.731\mathbf{i} + 1.371\mathbf{j} - 1.829\mathbf{k}) \\ &= -2.74\mathbf{i} + 4.39\mathbf{j} + 2.19\mathbf{k} \text{ kN}\cdot\text{m} \quad \textcircled{1}\end{aligned}$$

Ans.

- Magnitude of the Moment

$$M_O = \sqrt{2.74^2 + 4.39^2 + 2.19^2} = 5.62 \text{ kN}\cdot\text{m}$$

Ans.



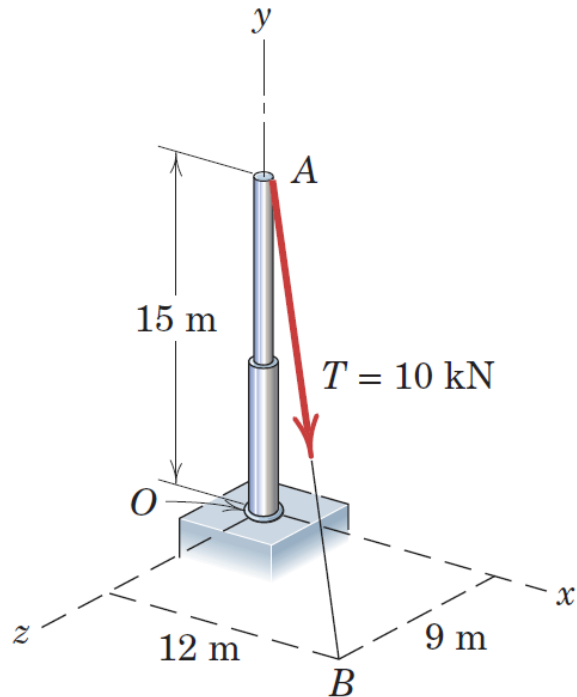
HELPFUL HINT

① The student should verify by inspection the signs of the moment components.

Article 2/8 – Sample Problem 2/13 (1 of 4)

- Problem Statement

A tension $\mathbf{T}$ of magnitude 10 kN is applied to the cable attached to the top A of the rigid mast and secured to the ground at B . Determine the moment M_z of $\mathbf{T}$ about the z -axis passing through the base O .

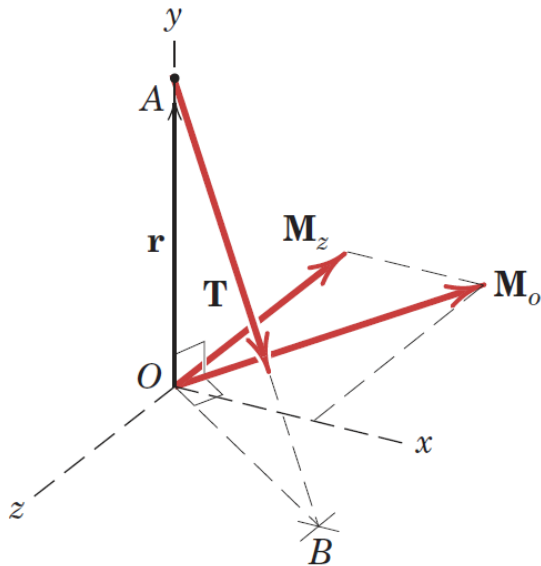


Article 2/8 – Sample Problem 2/13 (2 of 4)

- Tension Vector

$$\begin{aligned}\mathbf{T} &= T\mathbf{n}_{AB} = 10 \left[\frac{12\mathbf{i} - 15\mathbf{j} + 9\mathbf{k}}{\sqrt{(12)^2 + (-15)^2 + (9)^2}} \right] \\ &= 10(0.566\mathbf{i} - 0.707\mathbf{j} + 0.424\mathbf{k}) \text{ kN}\end{aligned}$$

- Solution 1 – Cross Product

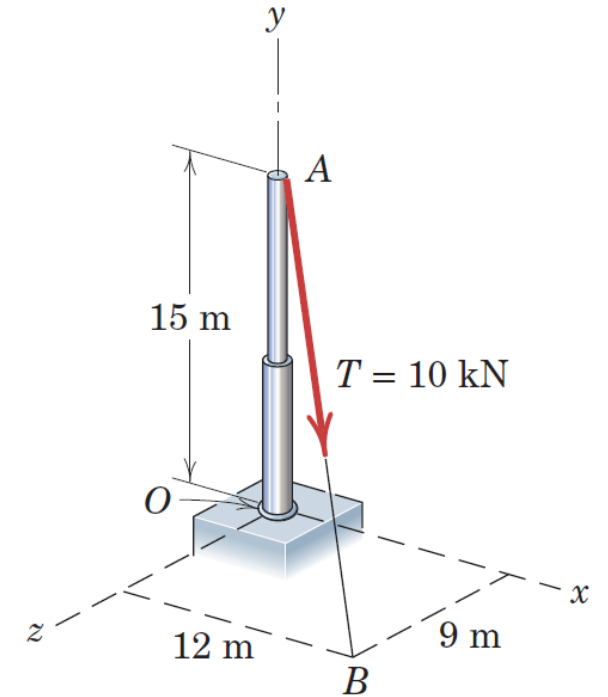


$$[\mathbf{M}_O = \mathbf{r} \times \mathbf{F}]$$

$$\begin{aligned}\mathbf{M}_O &= 15\mathbf{j} \times 10(0.566\mathbf{i} - 0.707\mathbf{j} + 0.424\mathbf{k}) \\ &= 150(-0.566\mathbf{k} + 0.424\mathbf{i}) \text{ kN}\cdot\text{m}\end{aligned}$$

The value M_z of the desired moment is the scalar component of $\mathbf{M}_O$ in the z -direction or $M_z = \mathbf{M}_O \cdot \mathbf{k}$. Therefore,

$$M_z = 150(-0.566\mathbf{k} + 0.424\mathbf{i}) \cdot \mathbf{k} = -84.9 \text{ kN}\cdot\text{m} \quad \text{Ans.}$$



Article 2/8 – Sample Problem 2/13 (3 of 4)

- Solution 2 – Two Scalar Components

$M_z = T_{xy}d$, where d is the perpendicular distance from T_{xy} to O . ③ The cosine of the angle between T and T_{xy} is $\sqrt{15^2 + 12^2}/\sqrt{15^2 + 12^2 + 9^2} = 0.906$, and therefore,

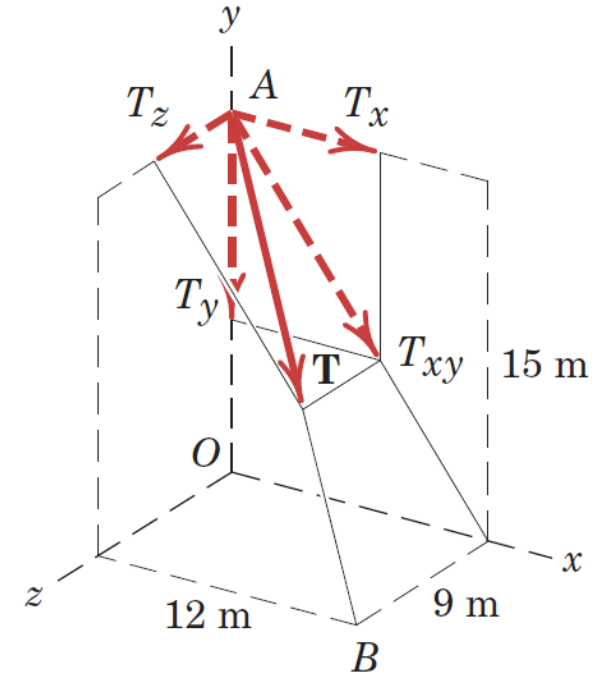
$$T_{xy} = 10(0.906) = 9.06 \text{ kN}$$

The moment arm d equals $\overline{OA}$ multiplied by the sine of the angle between T_{xy} and OA , or

$$d = 15 \frac{12}{\sqrt{12^2 + 15^2}} = 9.37 \text{ m}$$

Hence, the moment of $\mathbf{T}$ about the z -axis has the magnitude

$$M_z = 9.06(9.37) = 84.9 \text{ kN}\cdot\text{m} \quad \text{Ans.}$$



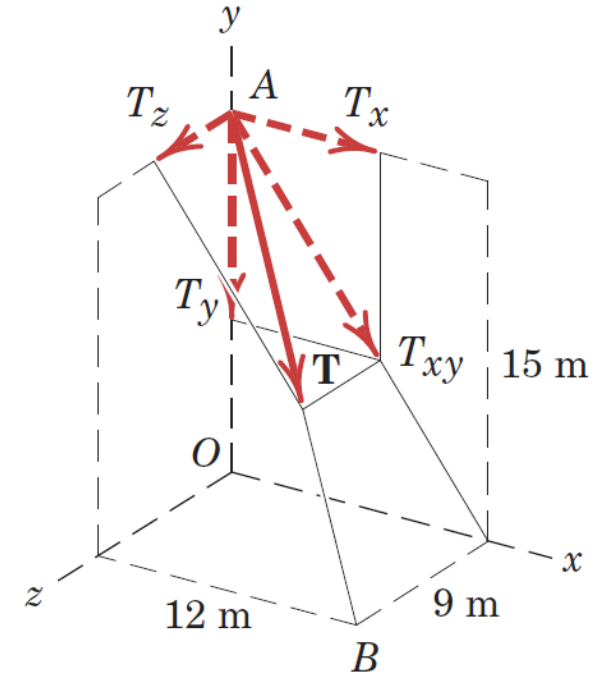
Article 2/8 – Sample Problem 2/13 (4 of 4)

- **Solution 3 – Three Scalar Components**

By inspection, only T_x makes a moment about point O . The y -component intersects point O and the z -component is parallel to the z -axis.

From before, $T_x = 10(0.566) = 5.66 \text{ kN}$.

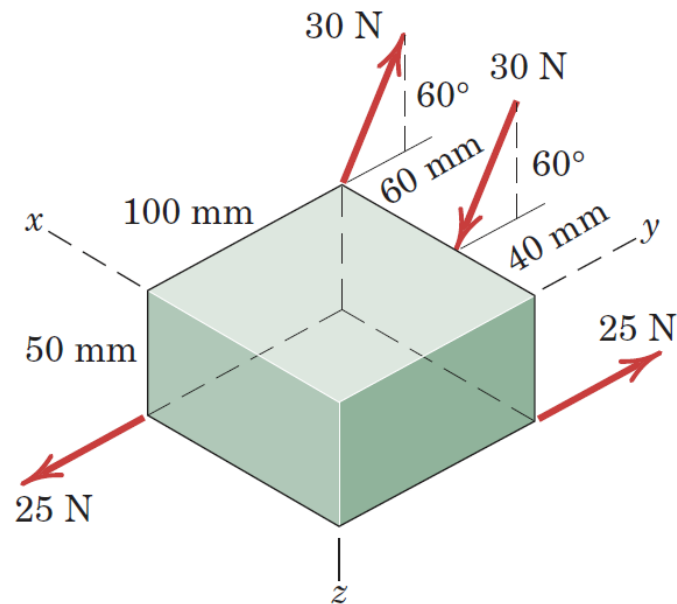
Therefore... $M_z = 5.66(15) = 84.9 \text{ kN}\cdot\text{m}$ *Ans.*



Article 2/8 – Sample Problem 2/14 (1 of 2)

- Problem Statement

Determine the magnitude and direction of the couple $\mathbf{M}$ which will replace the two given couples and still produce the same external effect on the block. Specify the two forces $\mathbf{F}$ and $-\mathbf{F}$, applied in the two faces of the block parallel to the y - z plane, which may replace the four given forces. The 30-N forces act parallel to the y - z plane.



Article 2/8 – Sample Problem 2/14 (2 of 2)

• Solution

The couple due to the 30-N forces has the magnitude $M_1 = 30(0.06) = 1.80 \text{ N}\cdot\text{m}$. The direction of $\mathbf{M}_1$ is normal to the plane defined by the two forces, and the sense, shown in the figure, is established by the right-hand convention. The couple due to the 25-N forces has the magnitude $M_2 = 25(0.10) = 2.50 \text{ N}\cdot\text{m}$ with the direction and sense shown in the same figure. The two couple vectors combine to give the components

$$M_y = 1.80 \sin 60^\circ = 1.559 \text{ N}\cdot\text{m}$$

$$M_z = -2.50 + 1.80 \cos 60^\circ = -1.600 \text{ N}\cdot\text{m}$$

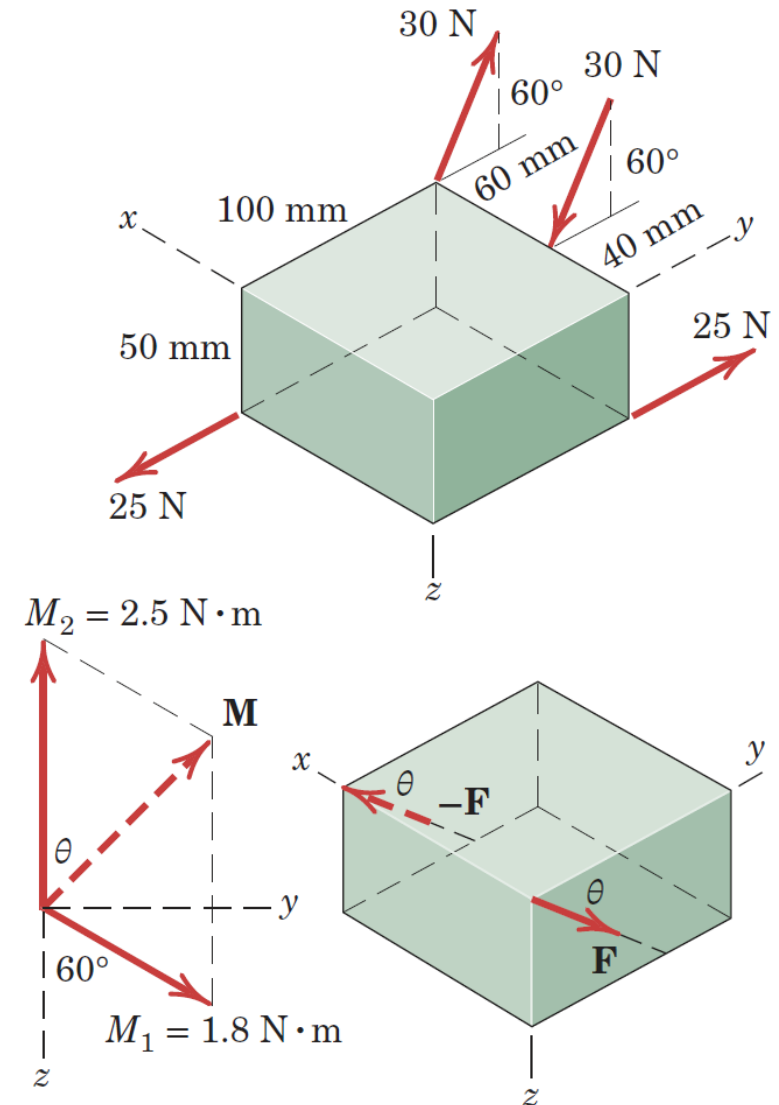
Thus, $M = \sqrt{(1.559)^2 + (-1.600)^2} = 2.23 \text{ N}\cdot\text{m}$ ① *Ans.*

with $\theta = \tan^{-1} \frac{1.559}{1.600} = \tan^{-1} 0.974 = 44.3^\circ$ *Ans.*

The forces $\mathbf{F}$ and $-\mathbf{F}$ lie in a plane normal to the couple $\mathbf{M}$, and their moment arm as seen from the right-hand figure is 100 mm. Thus, each force has the magnitude

$$[M = Fd] \quad F = \frac{2.23}{0.10} = 22.3 \text{ N} \quad \textit{Ans.}$$

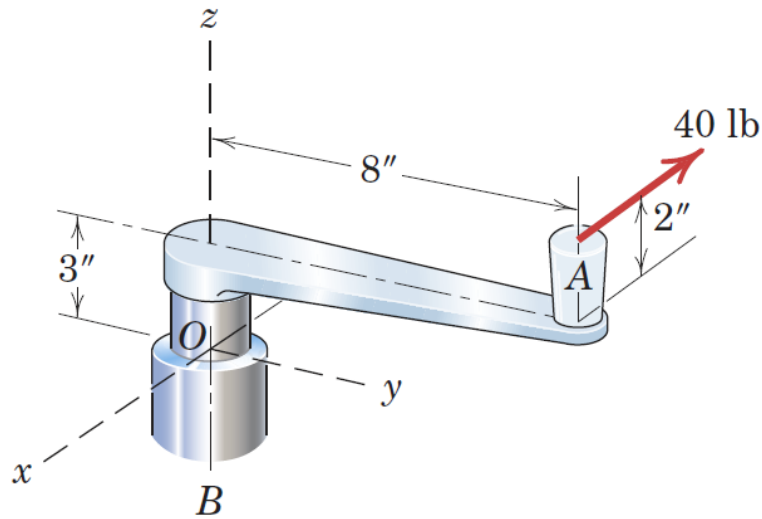
and the direction $\theta = 44.3^\circ$.



Article 2/8 – Sample Problem 2/15 (1 of 2)

- Problem Statement

A force of 40 lb is applied at A to the handle of the control lever which is attached to the fixed shaft OB . In determining the effect of the force on the shaft at a cross section such as that at O , we may replace the force by an equivalent force at O and a couple. Describe this couple as a vector $\mathbf{M}$.



Article 2/8 – Sample Problem 2/15 (2 of 2)

- Solution

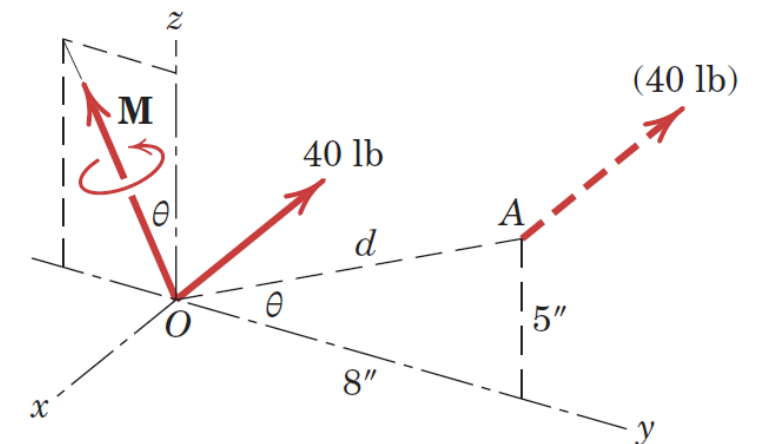
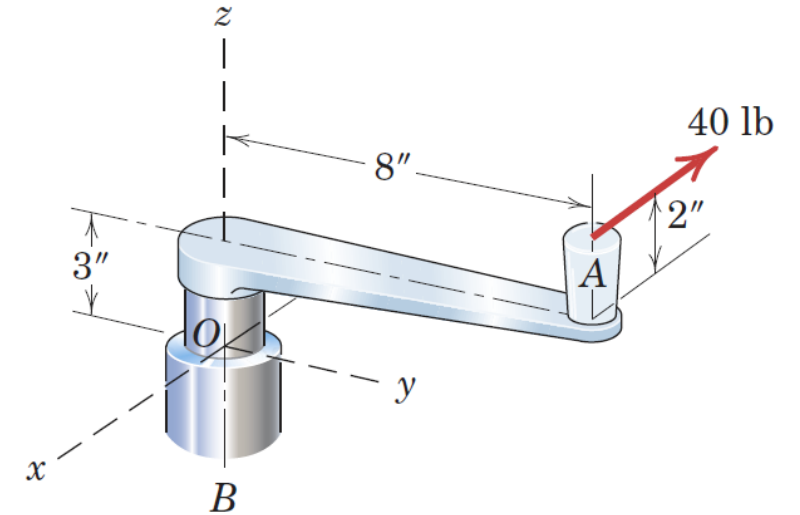
$$\mathbf{M} = (8\mathbf{j} + 5\mathbf{k}) \times (-40\mathbf{i}) = -200\mathbf{j} + 320\mathbf{k} \text{ lb-in.}$$

Alternatively we see that moving the 40-lb force through a distance $d = \sqrt{5^2 + 8^2} = 9.43$ in. to a parallel position through O requires the addition of a couple $\mathbf{M}$ whose magnitude is

$$M = Fd = 40(9.43) = 377 \text{ lb-in.} \quad \text{Ans.}$$

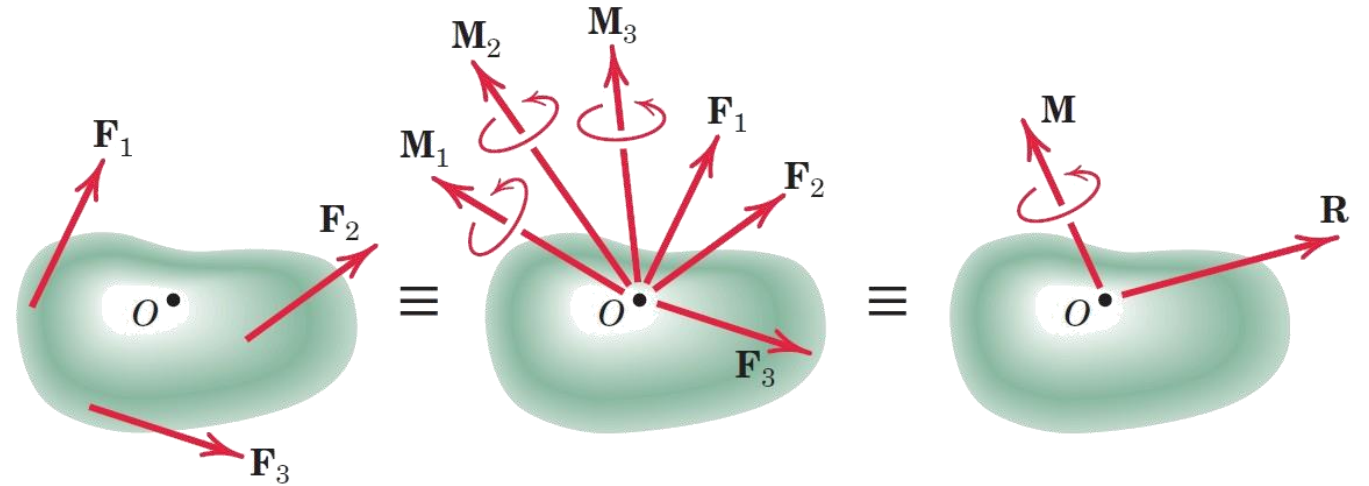
The couple vector is perpendicular to the plane in which the force is shifted, and its sense is that of the moment of the given force about O . The direction of $\mathbf{M}$ in the y - z plane is given by

$$\theta = \tan^{-1} \frac{5}{8} = 32.0^\circ \quad \text{Ans.}$$



Article 2/9 Resultants (3D)

- Illustration



- Equations of Interest

$$\mathbf{R} = \mathbf{F}_1 + \mathbf{F}_2 + \mathbf{F}_3 + \cdots = \Sigma \mathbf{F}$$

$$\mathbf{M} = \mathbf{M}_1 + \mathbf{M}_2 + \mathbf{M}_3 + \cdots = \Sigma (\mathbf{r} \times \mathbf{F})$$

$$R_x = \Sigma F_x \quad R_y = \Sigma F_y \quad R_z = \Sigma F_z$$

$$R = \sqrt{(\Sigma F_x)^2 + (\Sigma F_y)^2 + (\Sigma F_z)^2}$$

$$\mathbf{M}_x = \Sigma (\mathbf{r} \times \mathbf{F})_x \quad \mathbf{M}_y = \Sigma (\mathbf{r} \times \mathbf{F})_y \quad \mathbf{M}_z = \Sigma (\mathbf{r} \times \mathbf{F})_z$$

$$M = \sqrt{M_x^2 + M_y^2 + M_z^2}$$

Article 2/9 – Types of Force Systems (1 of 2)

- Concurrent Forces

- Because the forces are concurrent, there are no moments about the point of concurrency.
- $\mathbf{R} = \Sigma \mathbf{F}$

- Coplanar Forces

- Article 2/6 was devoted to this force system.

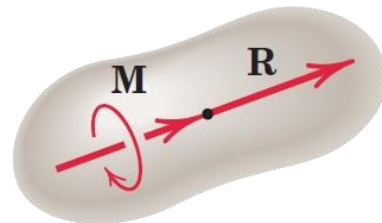
Article 2/9 – Types of Force Systems (2 of 2)

- Parallel Forces not in the Same Plane
 - Because the forces are parallel, the moment they produce about any point will be perpendicular to the line of action of the resultant.
- Calculation Steps
 1. Find the resultant, $\mathbf{R} = \Sigma \mathbf{F}$
 2. Find the couple at the point, $\mathbf{M}_O = \Sigma \mathbf{M}_O$ (from all forces or applied couples)
 3. Write a position vector $\mathbf{r}$ from the force-couple reference point to any point on the line of action of the resultant $\mathbf{R}$. Typically, the point will be specified in one of the three coordinate-axis planes.
 - a. For a point in the x-y plane $\mathbf{r} = x\mathbf{i} + y\mathbf{j}$
 - b. For a point in the x-z plane, $\mathbf{r} = x\mathbf{i} + z\mathbf{k}$
 - c. For a point in the y-z plane, $\mathbf{r} = y\mathbf{j} + z\mathbf{k}$
 4. Solve the equation $\mathbf{r} \times \mathbf{R} = \mathbf{M}_O$

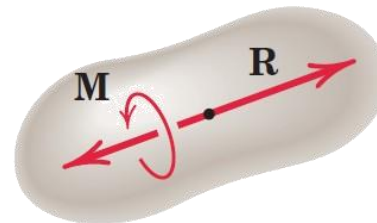
Article 2/9 – Wrench Resultants (1 of 4)

- Occurrence and Illustration

- Wrenches occur for a system of forces which are not parallel or concurrent. In this case, the resultant couple vector, $\mathbf{M}_O$, will have a component that is parallel to the resultant force $\mathbf{R}$. The resultant $\mathbf{R}$ is not able to produce this component of the moment regardless of its position relative to the force-couple system reference point.
- The simplified force system will consist of two pieces.
 - A resultant $\mathbf{R}$ which equals the vector sum of all forces, and is positioned such that it can produce the part of the resultant couple vector which is perpendicular to its line of action.
 - A wrench moment $\mathbf{M}$ which is equal to the part of the resultant couple vector which is parallel to the line of action of the resultant $\mathbf{R}$.



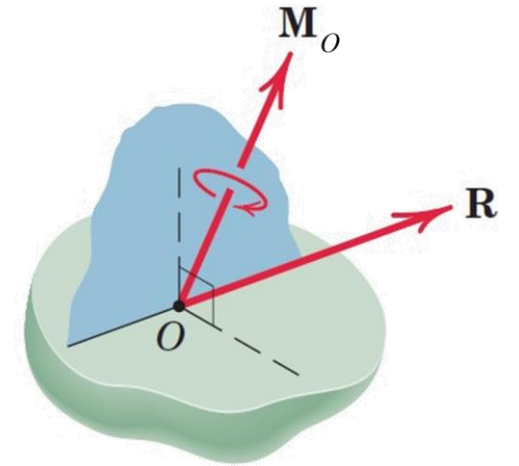
Positive wrench



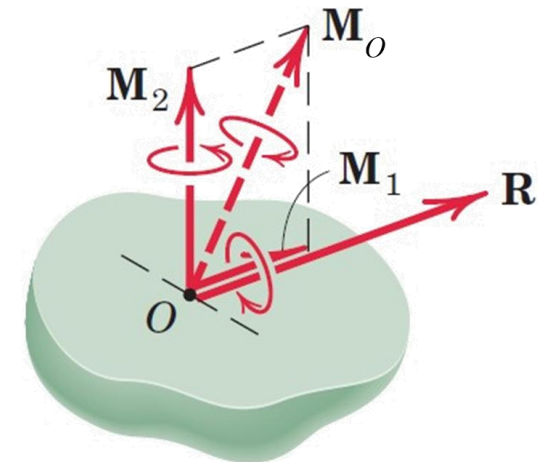
Negative wrench

Article 2/9 – Wrench Resultants (2 of 4)

- General Force-Couple System

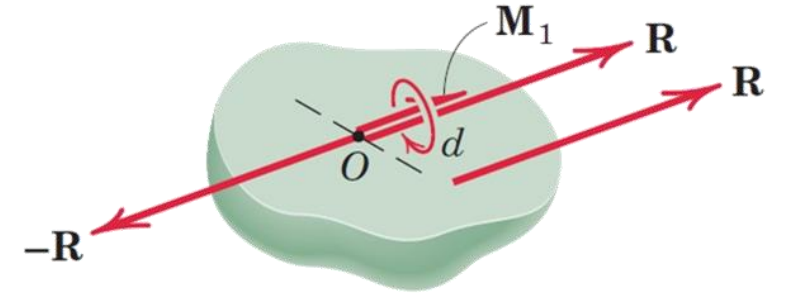


- Components of the Resultant Couple Vector
 - $\mathbf{M}_1$ is Parallel to $\mathbf{R}$
 - $\mathbf{M}_2$ is Perpendicular to $\mathbf{R}$

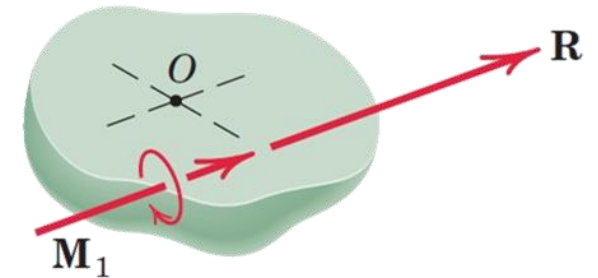


Article 2/9 – Wrench Resultants (3 of 4)

- Relocation of the Resultant by a Couple
 - $\mathbf{R}$ is moved a distance d from the reference point.
 - $Rd = M_2$



- Final Wrench Resultant
 - Force Resultant is Preserved
 - Moment Resultant is Preserved
 - $\mathbf{M}_1$ is Simply Added
 - $\mathbf{R}$ will Produce $\mathbf{M}_2$
 - $\mathbf{M}_O = \mathbf{M}_1 + \mathbf{M}_2$



Article 2/9 – Wrench Resultants (4 of 4)

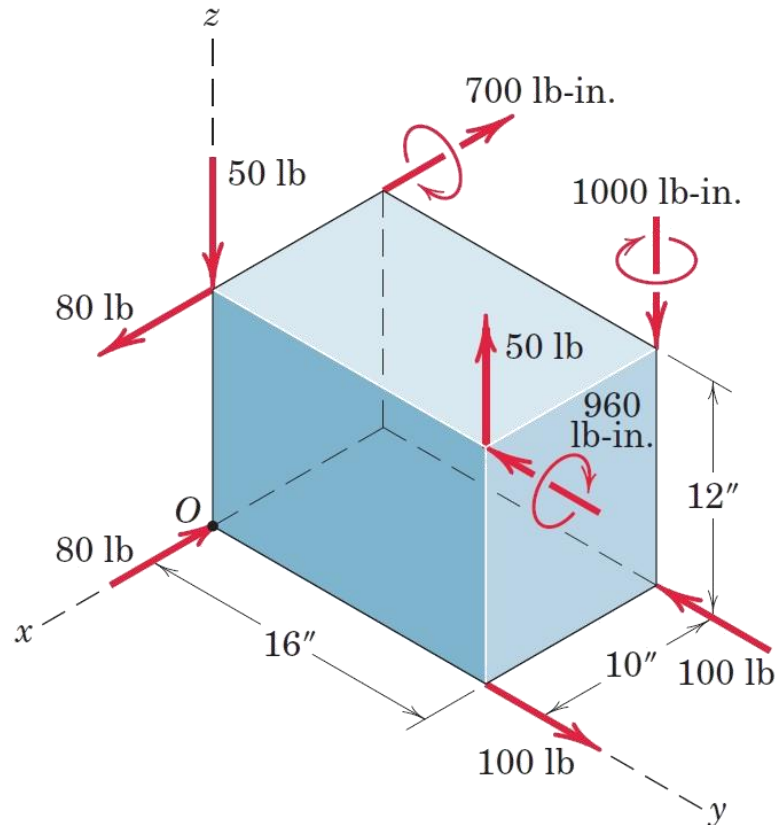
- Calculation Steps

1. Find the resultant, $\mathbf{R} = \Sigma \mathbf{F}$
2. Find the couple at the point, $\mathbf{M}_O = \Sigma \mathbf{M}_O$ (from all forces or applied couples)
3. Find the wrench moment, $\mathbf{M}_1$
 - a. Write a unit vector $\mathbf{n}$ in the direction of the resultant $\mathbf{R}$, $\mathbf{n} = \mathbf{R}/R$
 - b. Take a Dot Product to find the scalar portion of $\mathbf{M}_O$ in the direction of $\mathbf{R}$, $M_1 = \mathbf{M}_O \cdot \mathbf{n}$
 - c. The algebraic sign of M_1 will tell you if the wrench is in the positive or negative sense.
 - d. Write the wrench-moment vector, $\mathbf{M}_1 = M_1 \mathbf{n}$
4. Write a position vector $\mathbf{r}$ from the force-couple reference point to any point on the line of action of the resultant $\mathbf{R}$. Typically, the point will be specified in one of the three coordinate-axis planes.
 - a. For a point in the x-y plane $\mathbf{r} = x\mathbf{i} + y\mathbf{j}$
 - b. For a point in the x-z plane, $\mathbf{r} = x\mathbf{i} + z\mathbf{k}$
 - c. For a point in the y-z plane, $\mathbf{r} = y\mathbf{j} + z\mathbf{k}$
5. Solve the equation $\mathbf{r} \times \mathbf{R} + \mathbf{M}_1 = \mathbf{M}_O$

Article 2/9 – Sample Problem 2/16 (1 of 2)

- Problem Statement

Determine the resultant of the force and couple system which acts on the rectangular solid.



Article 2/9 – Sample Problem 2/16 (2 of 2)

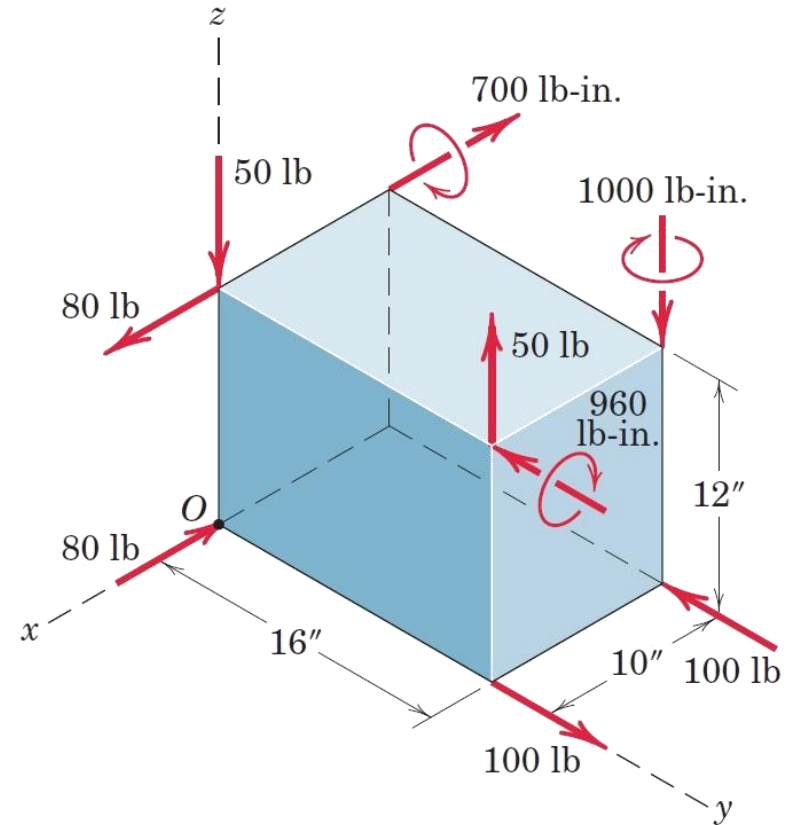
- Solution

$$\mathbf{R} = \Sigma \mathbf{F} = (80 - 80)\mathbf{i} + (100 - 100)\mathbf{j} + (50 - 50)\mathbf{k} = \mathbf{0} \text{ lb} \quad \textcircled{1}$$

The sum of the moments about O is

$$\begin{aligned} \mathbf{M}_O &= [50(16) - 700]\mathbf{i} + [80(12) - 960]\mathbf{j} + [100(10) - 1000]\mathbf{k} \text{ lb-in.} \\ &= 100\mathbf{i} \text{ lb-in.} \quad \textcircled{2} \end{aligned}$$

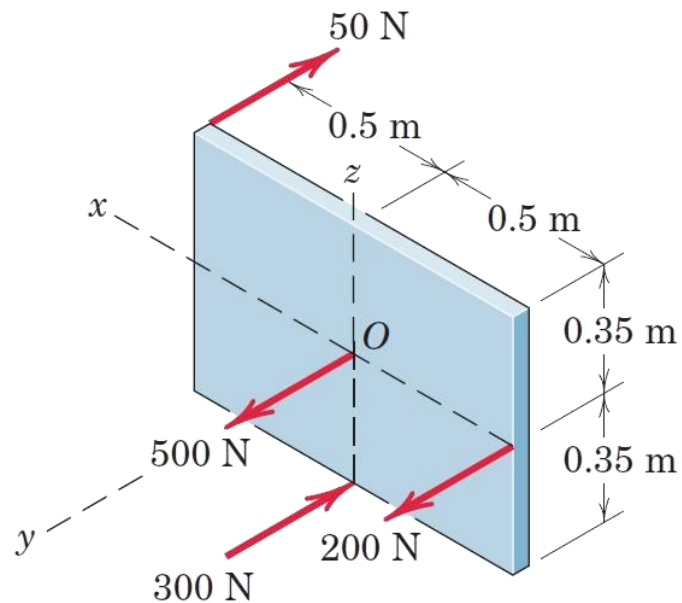
- ① Since the force summation is zero, we conclude that the resultant, if it exists, must be a couple.
- ② The moments associated with the force pairs are easily obtained by using the $M = Fd$ rule and assigning the unit-vector direction by inspection. In many three-dimensional problems, this may be simpler than the $\mathbf{M} = \mathbf{r} \times \mathbf{F}$ approach.



Article 2/9 – Sample Problem 2/17 (1 of 2)

- Problem Statement

Determine the resultant of the system of parallel forces which act on the plate. Solve with a vector approach.



Article 2/9 – Sample Problem 2/17 (2 of 2)

- Solution

$$\mathbf{R} = \Sigma \mathbf{F} = (200 + 500 - 300 - 50)\mathbf{j} = 350\mathbf{j} \text{ N}$$

$$\begin{aligned}\mathbf{M}_O &= [50(0.35) - 300(0.35)]\mathbf{i} + [-50(0.50) - 200(0.50)]\mathbf{k} \\ &= -87.5\mathbf{i} - 125\mathbf{k} \text{ N}\cdot\text{m}\end{aligned}$$

The placement of $\mathbf{R}$ so that it alone represents the above force–couple system is determined by the principle of moments in vector form

$$\mathbf{r} \times \mathbf{R} = \mathbf{M}_O$$

$$(x\mathbf{i} + y\mathbf{j} + z\mathbf{k}) \times 350\mathbf{j} = -87.5\mathbf{i} - 125\mathbf{k}$$

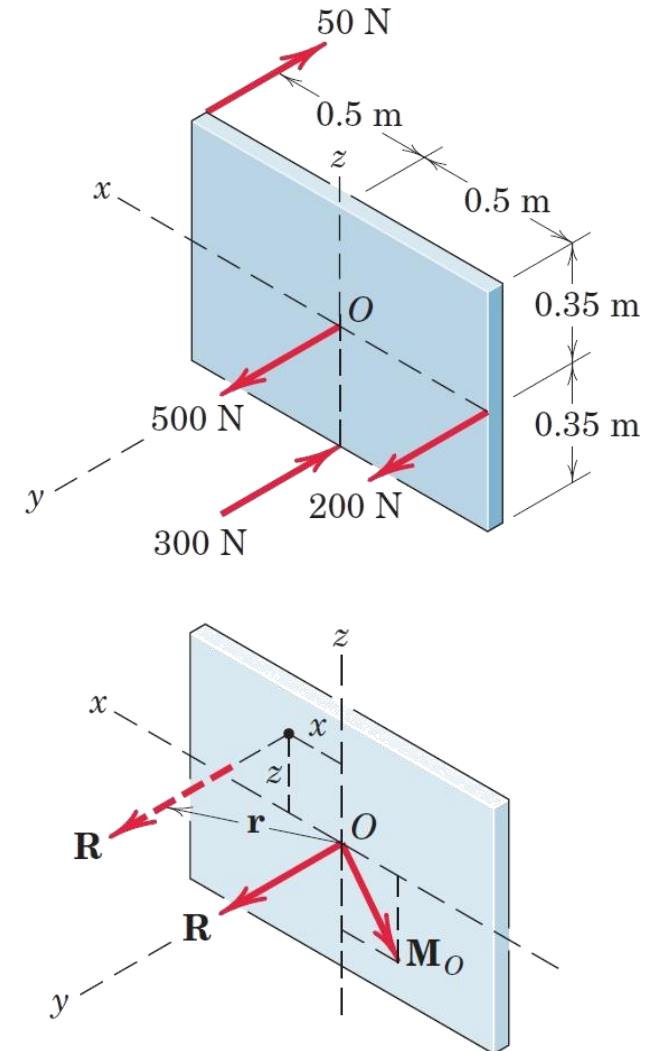
$$350x\mathbf{k} - 350z\mathbf{i} = -87.5\mathbf{i} - 125\mathbf{k}$$

From the one vector equation we may obtain the two scalar equations

$$350x = -125 \quad \text{and} \quad -350z = -87.5$$

Hence, $x = -0.357 \text{ m}$ and $z = 0.250 \text{ m}$ are the coordinates through which the line of action of $\mathbf{R}$ must pass. The value of y may, of course, be any value, as permitted by the principle of transmissibility. Thus, as expected, the variable y drops out of the above vector analysis. ①

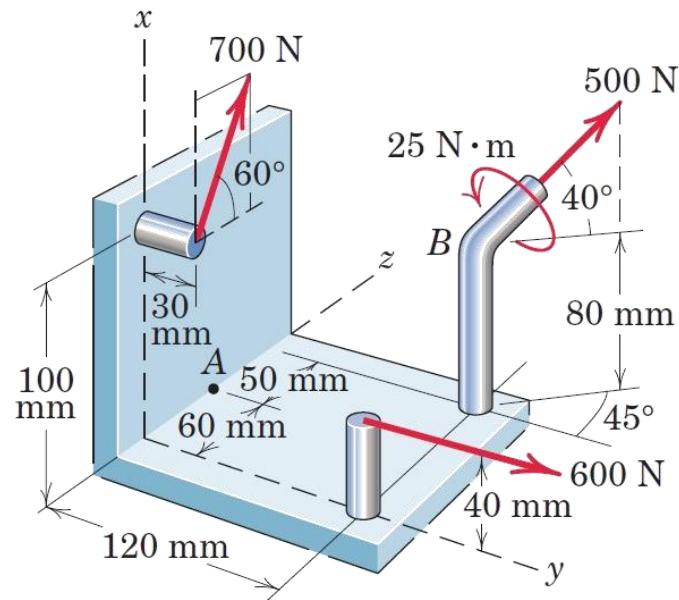
① You should also carry out a scalar solution to this problem.



Article 2/9 – Sample Problem 2/18 (1 of 4)

- Problem Statement

Replace the two forces and the negative wrench by a single force $\mathbf{R}$ applied at A and the corresponding couple $\mathbf{M}$.



Article 2/9 – Sample Problem 2/18 (2 of 4)

• Force Resultant

$$[R_x = \Sigma F_x] \quad R_x = 500 \sin 40^\circ + 700 \sin 60^\circ = 928 \text{ N}$$

$$[R_y = \Sigma F_y] \quad R_y = 600 + 500 \cos 40^\circ \cos 45^\circ = 871 \text{ N}$$

$$[R_z = \Sigma F_z] \quad R_z = 700 \cos 60^\circ + 500 \cos 40^\circ \sin 45^\circ = 621 \text{ N}$$

Thus, $\mathbf{R} = 928\mathbf{i} + 871\mathbf{j} + 621\mathbf{k} \text{ N}$

and $R = \sqrt{(928)^2 + (871)^2 + (621)^2} = 1416 \text{ N}$ *Ans.*

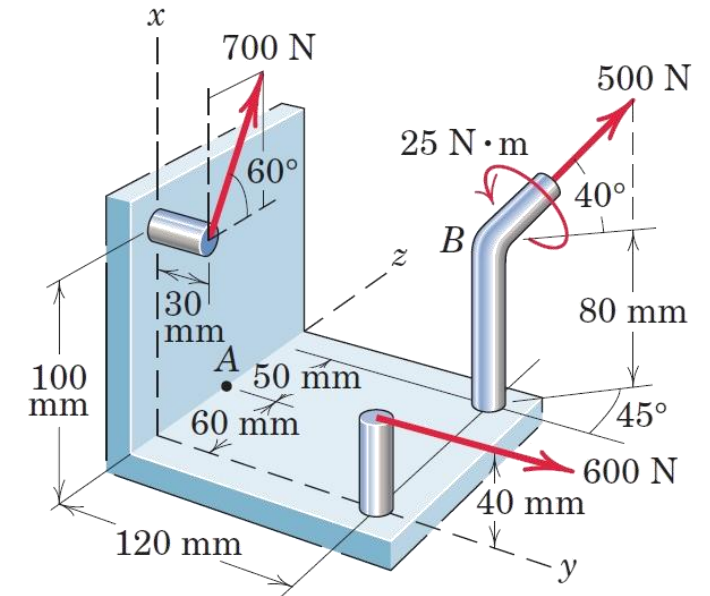
• Couple from the 500-N Force

$$[\mathbf{M} = \mathbf{r} \times \mathbf{F}] \quad \mathbf{M}_{500} = (0.08\mathbf{i} + 0.12\mathbf{j} + 0.05\mathbf{k}) \times 500(\mathbf{i} \sin 40^\circ + \mathbf{j} \cos 40^\circ \cos 45^\circ + \mathbf{k} \cos 40^\circ \sin 45^\circ) \quad \textcircled{1}$$

where $\mathbf{r}$ is the vector from A to B .

The term-by-term, or determinant, expansion gives

$$\mathbf{M}_{500} = 18.95\mathbf{i} - 5.59\mathbf{j} - 16.90\mathbf{k} \text{ N}\cdot\text{m}$$



① *Suggestion:* Check the cross-product results by evaluating the moments about A of the components of the 500-N force directly from the sketch.

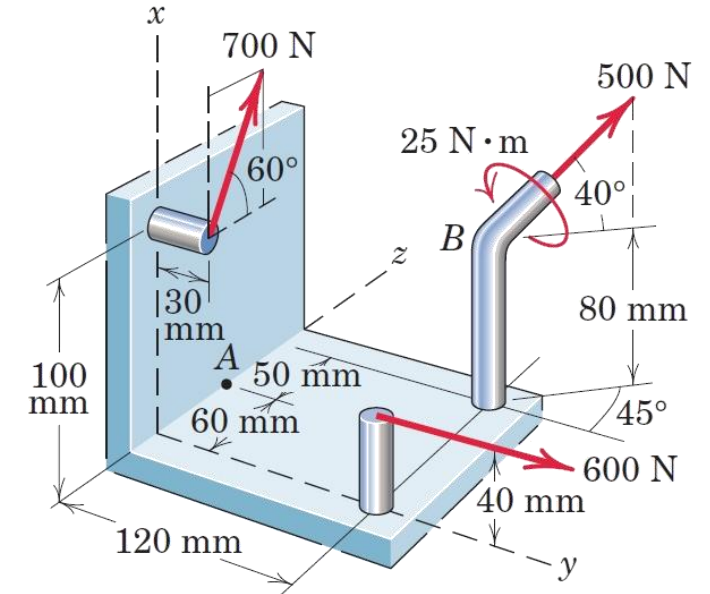
Article 2/9 – Sample Problem 2/18 (3 of 4)

- Couple from the 600-N Force

$$\begin{aligned}\mathbf{M}_{600} &= (600)(0.060)\mathbf{i} + (600)(0.040)\mathbf{k} \\ &= 36.0\mathbf{i} + 24.0\mathbf{k} \text{ N}\cdot\text{m}\end{aligned}$$

- Couple from the 700-N Force

$$\begin{aligned}\mathbf{M}_{700} &= (700 \cos 60^\circ)(0.030)\mathbf{i} - [(700 \sin 60^\circ)(0.060) \\ &\quad + (700 \cos 60^\circ)(0.100)]\mathbf{j} - (700 \sin 60^\circ)(0.030)\mathbf{k} \\ &= 10.5\mathbf{i} - 71.4\mathbf{j} - 18.19\mathbf{k} \text{ N}\cdot\text{m}\end{aligned}$$



② For the 600-N and 700-N forces it is easier to obtain the components of their moments about the coordinate directions through A by inspection of the figure than it is to set up the cross-product relations.

Article 2/9 – Sample Problem 2/18 (4 of 4)

- Couple from the Wrench Moment

$$\begin{aligned}\mathbf{M}' &= 25.0(-\mathbf{i} \sin 40^\circ - \mathbf{j} \cos 40^\circ \cos 45^\circ - \mathbf{k} \cos 40^\circ \sin 45^\circ) \\ &= -16.07\mathbf{i} - 13.54\mathbf{j} - 13.54\mathbf{k} \text{ N}\cdot\text{m}\end{aligned}$$

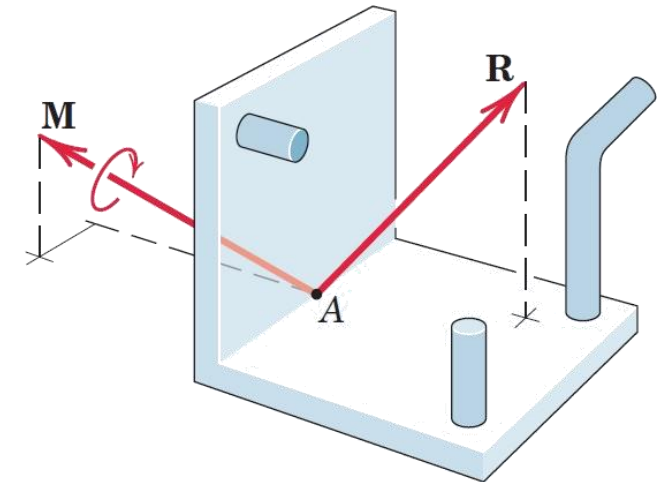
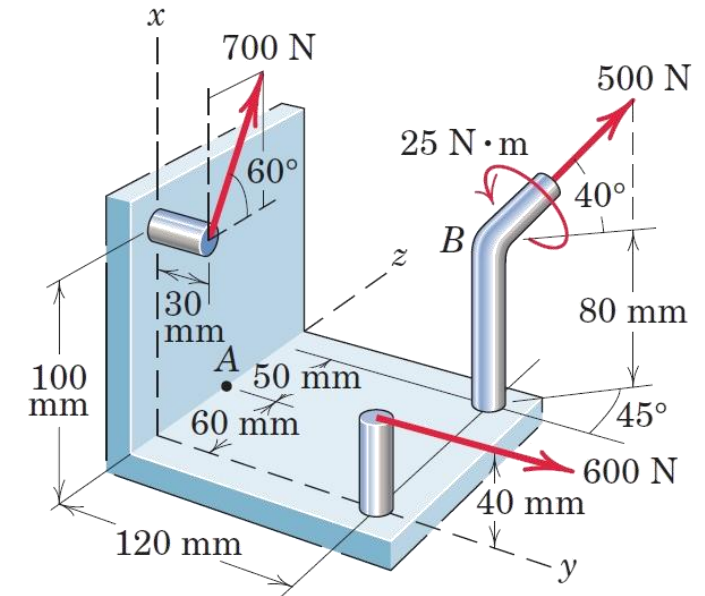
- Resultant Couple at A

$$\mathbf{M} = 49.4\mathbf{i} - 90.5\mathbf{j} - 24.6\mathbf{k} \text{ N}\cdot\text{m} \quad \textcircled{4}$$

and $M = \sqrt{(49.4)^2 + (90.5)^2 + (24.6)^2} = 106.0 \text{ N}\cdot\text{m}$ Ans.

③ The 25-N·m couple vector of the *wrench* points in the direction opposite to that of the 500-N force, and we must resolve it into its *x*-, *y*-, and *z*-components to be added to the other couple-vector components.

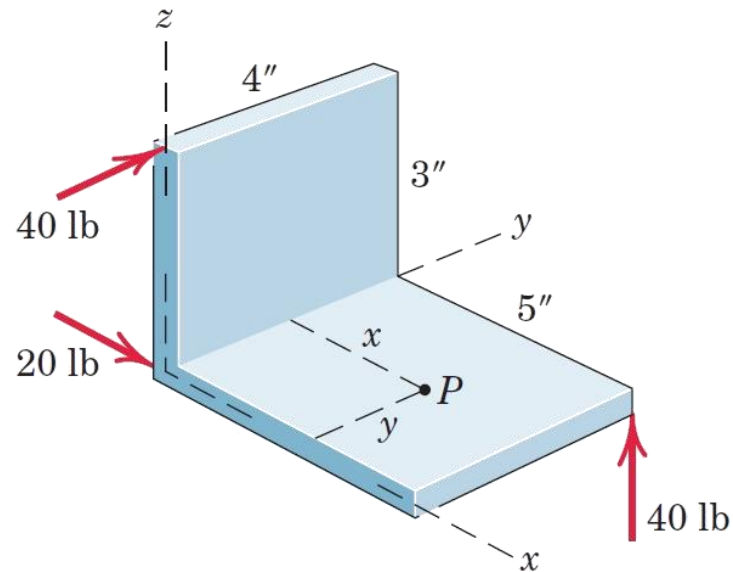
④ Although the resultant couple vector $\mathbf{M}$ in the sketch of the resultants is shown through A, we recognize that a couple vector is a free vector and therefore has no specified line of action.



Article 2/9 – Sample Problem 2/19 (1 of 3)

- Problem Statement

Determine the wrench resultant of the three forces acting on the bracket. Calculate the coordinates of the point P in the x - y plane through which the resultant force of the wrench acts. Also find the magnitude of the couple $\mathbf{M}$ of the wrench.



Article 2/9 – Sample Problem 2/19 (2 of 3)

- Resultant Force

$$\mathbf{R} = 20\mathbf{i} + 40\mathbf{j} + 40\mathbf{k} \text{ lb} \quad R = \sqrt{(20)^2 + (40)^2 + (40)^2} = 60 \text{ lb}$$

and its direction cosines are

$$\cos \theta_x = 20/60 = 1/3 \quad \cos \theta_y = 40/60 = 2/3 \quad \cos \theta_z = 40/60 = 2/3$$

- Moment about P

The moment of the wrench couple must equal the sum of the moments of the given forces about point P through which $\mathbf{R}$ passes. The moments about P of the three forces are

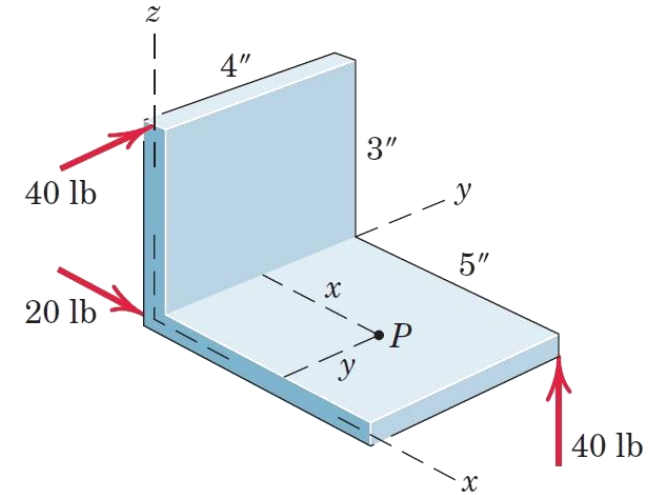
$$(\mathbf{M})_{R_x} = 20y\mathbf{k} \text{ lb-in.}$$

$$(\mathbf{M})_{R_y} = -40(3)\mathbf{i} - 40x\mathbf{k} \text{ lb-in.}$$

$$(\mathbf{M})_{R_z} = 40(4 - y)\mathbf{i} - 40(5 - x)\mathbf{j} \text{ lb-in.}$$

and the total moment is

$$\mathbf{M} = (40 - 40y)\mathbf{i} + (-200 + 40x)\mathbf{j} + (-40x + 20y)\mathbf{k} \text{ lb-in.}$$



Article 2/9 – Sample Problem 2/19 (3 of 3)

- Final Solution

The direction cosines of $\mathbf{M}$ are

$$\cos \theta_x = (40 - 40y)/M$$

$$\cos \theta_y = (-200 + 40x)/M$$

$$\cos \theta_z = (-40x + 20y)/M$$

where M is the magnitude of $\mathbf{M}$. Equating the direction cosines of $\mathbf{R}$ and $\mathbf{M}$ gives

$$40 - 40y = \frac{M}{3}$$

$$-200 + 40x = \frac{2M}{3}$$

$$-40x + 20y = \frac{2M}{3}$$

Solution of the three equations gives

$$M = -120 \text{ lb-in.} \quad x = 3 \text{ in.} \quad y = 2 \text{ in.} \quad \text{Ans.}$$

