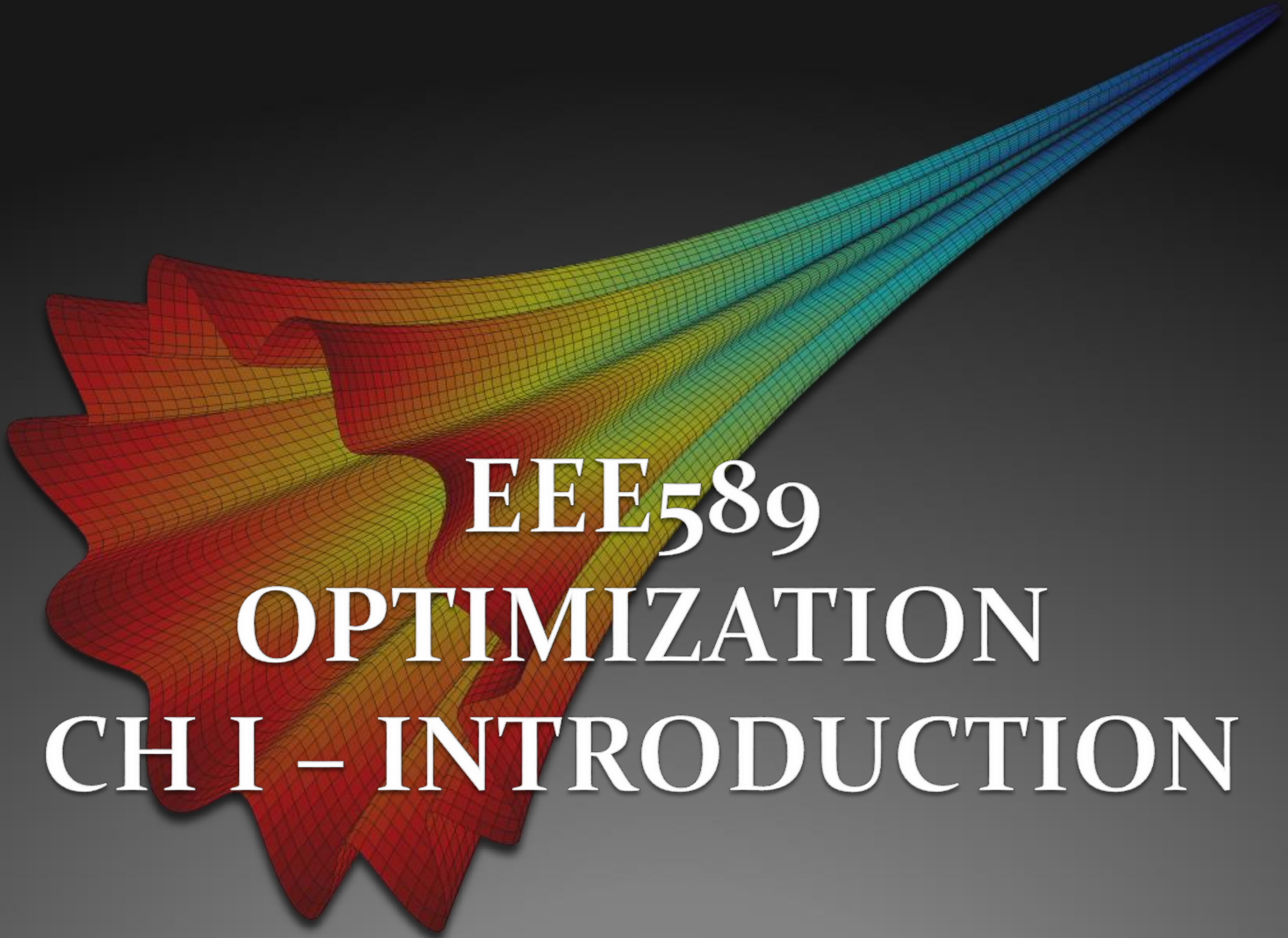
EEE589
OPTIMIZATION
CH I – INTRODUCTION



Course Information

- **Name of the Course:** OPTIMIZATION
- **Lecturer:** Dr. Seydi Kaçmaz
- **Office:** 201 , **E-mail:** seydikacmaz@gantep.edu.tr
- **Web Announcements:** https://akbis.gantep.edu.tr/detay/?A_ID=149283_doktor-ogretim-uyesi_seydi-kacmaz

Course Layout

- Prerequisites

    Linear algebra (matrices, vectors)

    Basic calculus

    Signal processing

    Matlab

- Text Book and References:

    1- Algorithms for Optimization (The MIT Press), Tim A. Wheeler, Mykel J. Kochenderfer, 2019

    2- S. Boyd and L. Vandenberghe, *Convex Optimization*, Cambridge University Press, 2004. (available at http://www.stanford.edu/~boyd/cvxbook/)

Course Content

- Introduction
- Derivatives and Gradients
- Bracketing
- Local Descent
- First-Order Methods
- Second-Order Methods
- Direct Methods
- Stochastic Methods
- Population Methods
- Constraints
- Linear Constrained Optimization
- Multiobjective Optimization
- Sampling Plans
- Surrogate Models
- Probabilistic Surrogate Models
- Surrogate Optimization
- Optimization under Uncertainty
- Uncertainty Propagation
- Discrete Optimization
- Expression Optimization
- Multidisciplinary Optimization

Grading

Homeworks	: %30
Project (Journal Implementation)	: %20
Presentation	: %10
Written Final Exam	: %40

Introduction

- Many disciplines involve optimization at their core.
  - In physics, systems are driven to their lowest energy state subject to physical laws.
  - In business, corporations aim to maximize shareholder value.
  - In biology, fitter organisms are more likely to survive.
- This lecture will focus on optimization from an engineering perspective, where the objective is to design a system that optimizes a set of metrics subject to constraints.
- The system could be a complex physical system like an aircraft, or it could be a simple structure such as a bicycle frame. The system might not even be physical; for example, we might be interested in designing a control system for an automated vehicle or a computer vision system that detects whether an image of a tumor biopsy is cancerous. We want these systems to perform as well as possible.
- Depending on the application, relevant metrics might include efficiency, safety, and accuracy.
- Constraints on the design might include cost, weight, and structural soundness.

Introduction

- Optimization is the act of obtaining the best result under given circumstances.

- Optimization can be defined as the process of finding the conditions that give the maximum or minimum of a function.

- The optimum seeking methods are also known as *mathematical programming techniques* and are generally studied as a part of operations research.

- *Operations research* is a branch of mathematics concerned with the application of scientific methods and techniques to decision making problems and with establishing the best or optimal solutions.

Introduction

- Applications of Optimization
  - Physics
  - Business
  - Biology
  - Engineering
- Objectives to Optimize
  - Efficiency
  - Safety
  - Accuracy
- Constraints
  - Cost
  - Weight
  - Structural Integrity
- Challenges
  - High-Dimensional Search Spaces
  - Multiple Competing Objectives
  - Model Uncertainty

Introduction

- **Operations research** (in the UK) or **operational research (OR)** (in the US) or **yöneylem araştırması** (in Turkish) is an interdisciplinary branch of mathematics which uses methods like:
  - mathematical modeling
  - statistics
  - algorithms to arrive at optimal or good decisions in complex problems which are concerned with optimizing the maxima (profit, faster assembly line, greater crop yield, higher bandwidth, etc) or minima (cost loss, lowering of risk, etc) of some objective function.
- The eventual intention behind using operations research is to elicit a best possible solution to a problem mathematically, which improves or optimizes the performance of the system.

A History

- Isaac Newton (1642-1727)
  (The development of differential calculus methods of optimization)

- Joseph-Louis Lagrange (1736-1813)
  (Calculus of variations, minimization of functionals,
  method of optimization for constrained problems)

- Augustin-Louis Cauchy (1789-1857)
  (Solution by direct substitution, steepest
  descent method for unconstrained optimization)

A History

- Leonhard Euler (1707-1783)
  (Calculus of variations, minimization of functionals)

- Gottfried Leibnitz (1646-1716)
  (Differential calculus methods of optimization)

İsim: Gottfired Wilhelm von Leibniz

A History

- George Bernard Dantzig (1914-2005)
  (Linear programming and Simplex method (1947))

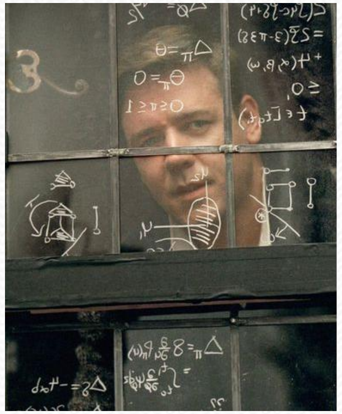
- Richard Bellman (1920-1984)
  (Principle of optimality in dynamic programming problems)

- Harold William Kuhn (1925-)
  (Necessary and sufficient conditions for the optimal solution of programming problems, game theory)

A History

- Albert William Tucker (1905-1995)
  (Necessary and sufficient conditions
  for the optimal solution of programming problems, nonlinear programming, game theory: his PhD student
  was John Nash)

- Von Neumann (1903-1957)
  (game theory)



John von Neumann

A History

- Calculus

The concept of calculus, or the study of continuous change, plays an important role in our discussion of optimization. Both differential and integral calculus make use of the notion of convergence of infinite series to a well-defined limit.

- Numerical Algorithms

The mid-twentieth century saw the rise of the electronic computer, spurring interest in numerical algorithms for optimization. The ease of calculations allowed optimization to be applied to much larger problems in a variety of domains.

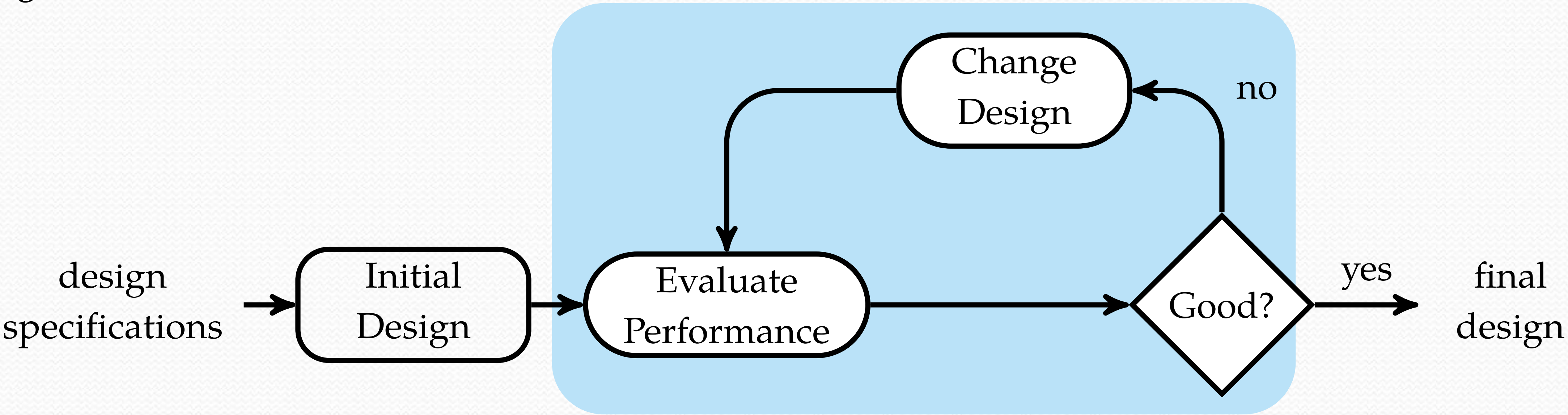
- Artificial Intelligence

Decades of advances in large scale computation have resulted in innovative physical engineering designs as well as the design of artificially intelligent systems.

- The optimization of deep neural networks is fueling a major revolution in artificial intelligence that will likely continue.

Optimization Process

A typical engineering design optimization process is shown in figure. The role of the designer is to provide a problem specification that details the parameters, constants, objectives, and constraints that are to be achieved. The designer is responsible for crafting the problem and quantifying the merits of potential designs. The designer also typically supplies a baseline design or initial design point to the optimization algorithm.



Optimization Process

- Optimization is about automating the process of refining the design to improve performance.
- An optimization algorithm is used to incrementally improve the design until it can no longer be improved or until the budgeted time or cost has been reached.
- The designer is responsible for analyzing the result of the optimization process to ensure its suitability for the final application.
- Misspecifications in the problem, poor baseline designs, and improperly implemented or unsuitable optimization algorithms can all lead to suboptimal or dangerous designs.

Basic Optimization Problem

$$\underset{\mathbf{x}}{\text{minimize}} \quad f(\mathbf{x})$$
$$\text{subject to} \quad \mathbf{x} \in \mathcal{X}$$

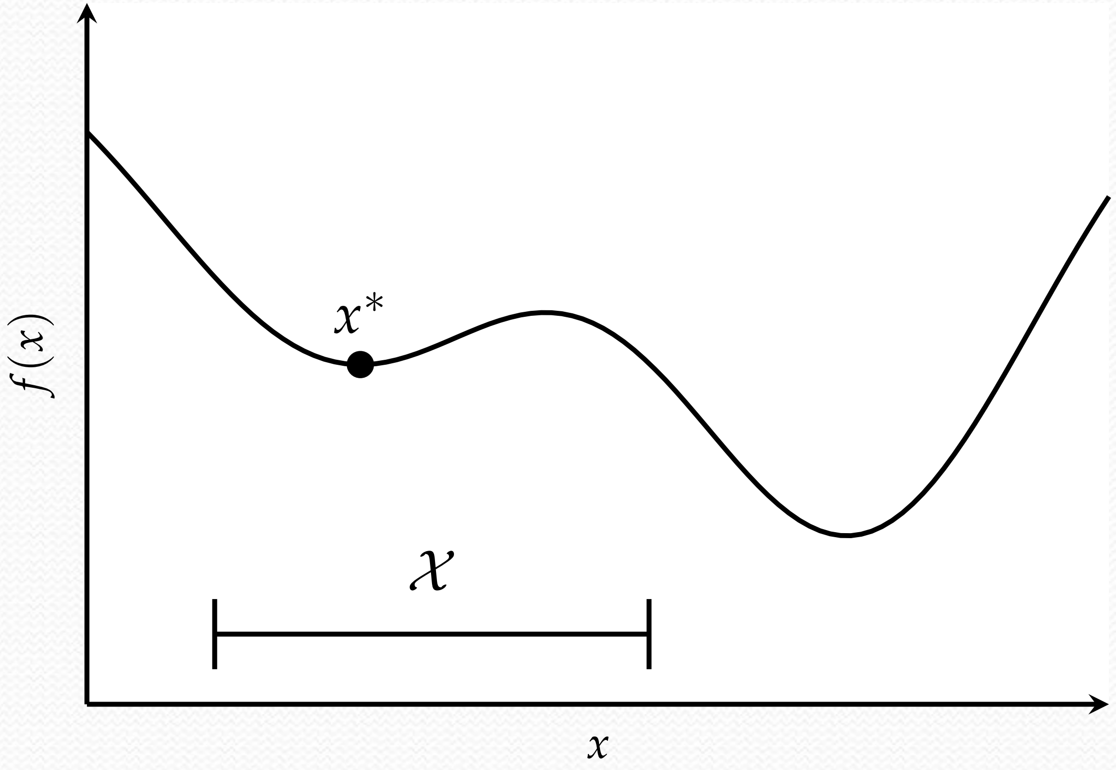
- x is a Design Point
- Design Variables $[x_1, x_2, \cdots, x_n]$
- Objective Function f
- Feasible Set $\mathcal{X}$
- A solution or Minimizer

Any value of x from among all points in the feasible set X that minimizes the objective function is called a solution or minimizer. A particular solution is written x^*.

Basic Optimization Problem

Figure shows an example of
a one-dimensional optimization problem.
Note that the minimum is merely the best
in the feasible set—lower points may exist
outside the feasible region.

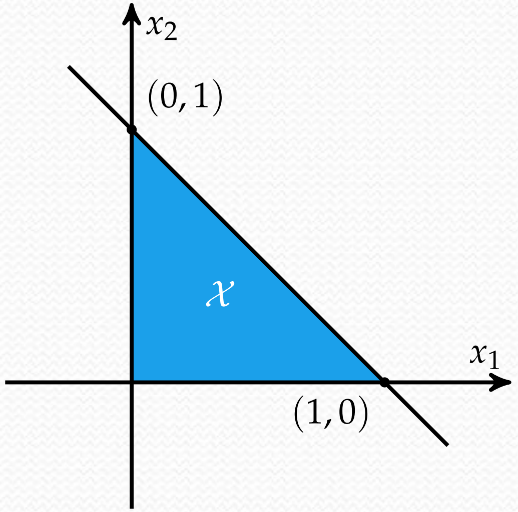
$$\underset{\mathbf{x}}{\text{minimize}} \quad f(\mathbf{x})$$
$$\text{subject to} \quad \mathbf{x} \in \mathcal{X}$$



Constraints

- Many problems have constraints. Each *constraint* limits the set of possible solutions, and together the constraints define the feasible set X. Feasible design points do not violate any constraints. For example, consider the following optimization problem and the feasible set that is plotted in figure.

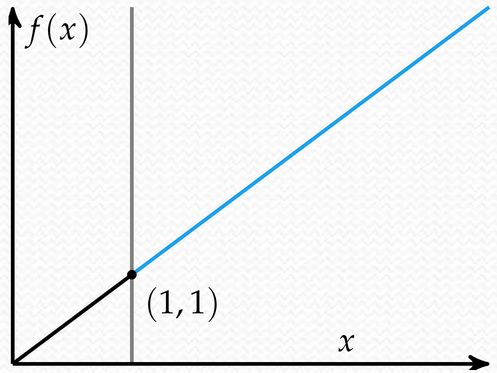
$$\begin{aligned} \underset{x_1, x_2}{\text{minimize}} \quad & f(x_1, x_2) \\ \text{subject to} \quad & x_1 \geq 0 \\ & x_2 \geq 0 \\ & x_1 + x_2 \leq 1 \end{aligned}$$



Constraints

- Constraints are typically written with ≤, ≥, or =. If constraints involve < or > (i.e., strict inequalities), then the feasible set does not include the constraint boundary. A potential issue with not including the boundary is illustrated by this problem and the feasible set that is shown in figure

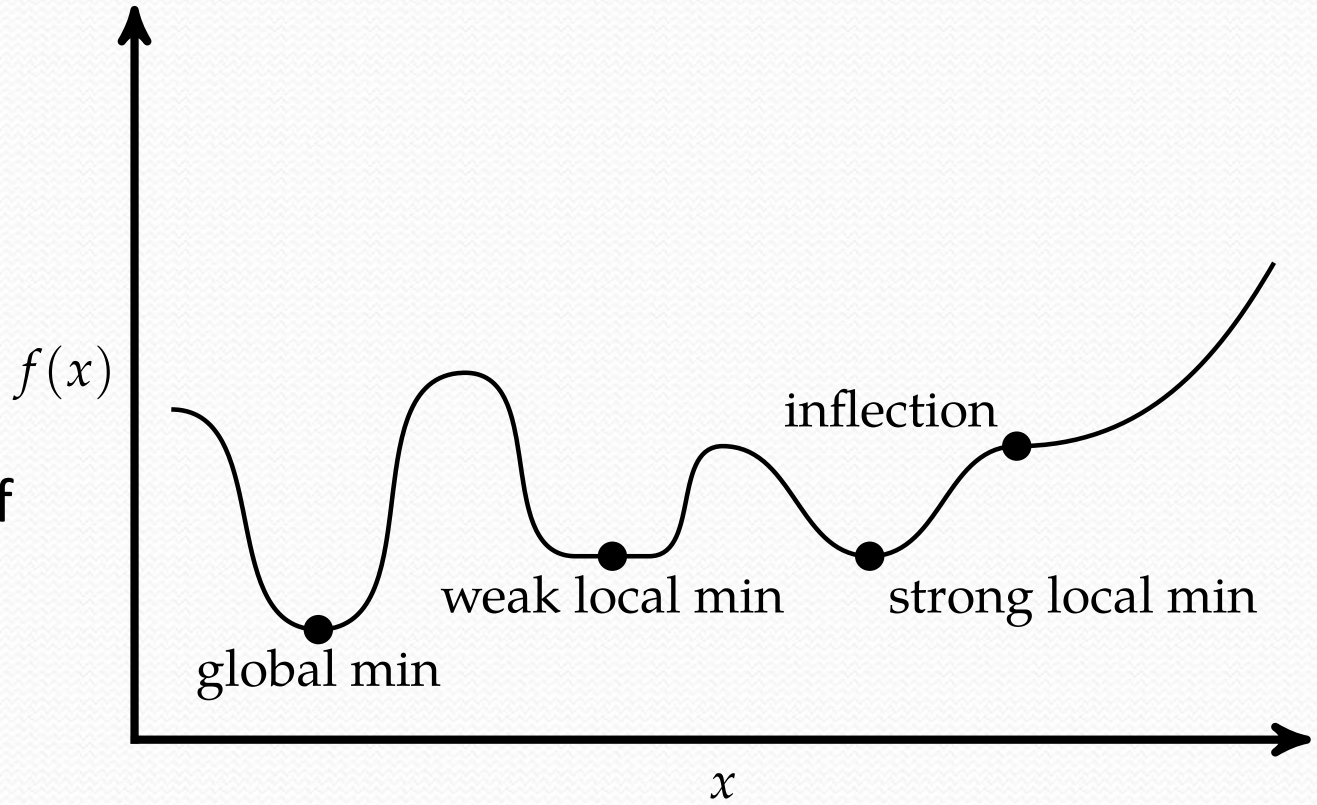
$$\begin{aligned} \underset{x}{\text{minimize}} \quad & x \\ \text{subject to} \quad & x > 1 \end{aligned}$$



The point x = 1 produces values smaller than any x greater than 1, but x = 1 is not feasible. We can pick any x arbitrarily close to, but greater than, 1, but no matter what we pick, we can always find an infinite number of values even closer to 1. We must conclude that the problem has no solution. To avoid such issues, it is often best to include the constraint boundary in the feasible set.

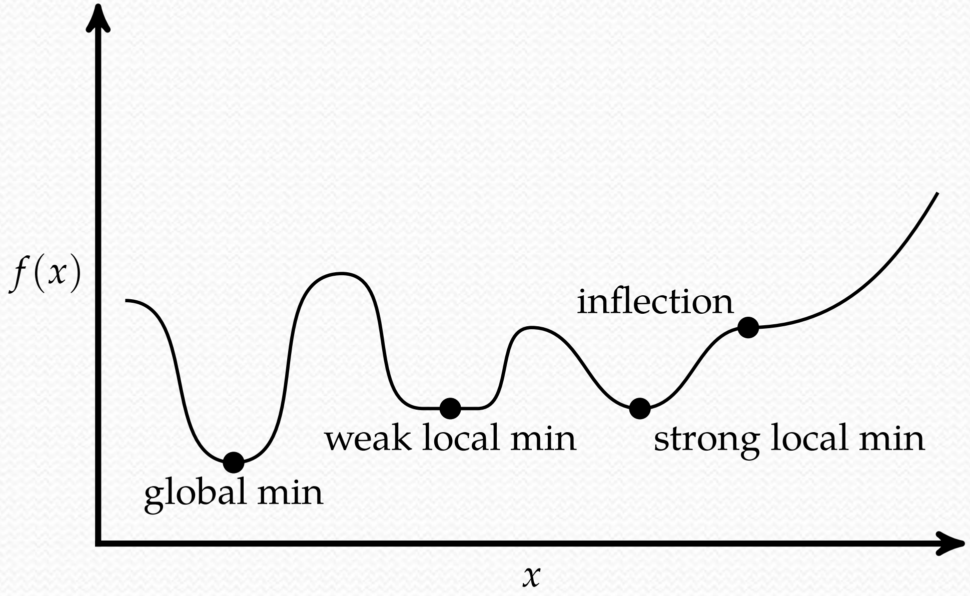
Critical Points

- Univariate Function
  - Figure shows a univariate function f(x) with several labeled critical points,(The term univariate describes objects involving one variable.) where the derivative is zero, that are of interest when discussing optimization problems.

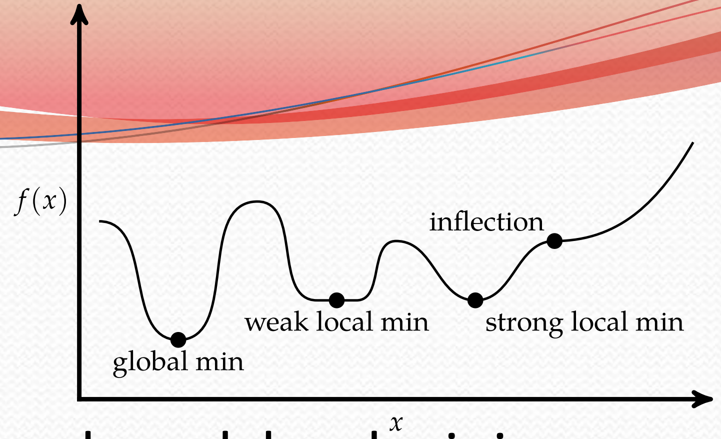


Critical Points

- Univariate Function
  - When minimizing f , we wish to find a global minimizer, a value of x for which f (x) is minimized. A function may have at most one global minimum, but it may have multiple global minimizers.
  - Unfortunately, it is generally difficult to prove that a given candidate point is at a global minimum. Often, the best we can do is check whether it is at a local minimum.



Critical Points



- Figure shows two types of local minima: strong local minima and weak local minima. A strong local minimizer, also known as a strict local minimizer, is a point that uniquely minimizes f within a neighborhood. A weak local minimizer is a local minimizer that is not a strong local minimizer.
- The derivative is zero at all local and global minima of continuous, unbounded objective functions. While having a zero derivative is a necessary condition for a local minimum, it is not a sufficient condition.
- Figure also has an inflection point where the derivative is zero but the point does not locally minimize f . An inflection point is where the sign of the second derivative of f changes, which corresponds to a local minimum or maximum of f ′. An inflection point does not necessarily have a zero derivative.

Conditions for Local Minima

- Many numerical optimization methods seek local minima. Local minima are locally optimal, but we do not generally know whether a local minimum is a global minimum. The conditions we discuss in this section assume that the objective function is differentiable.

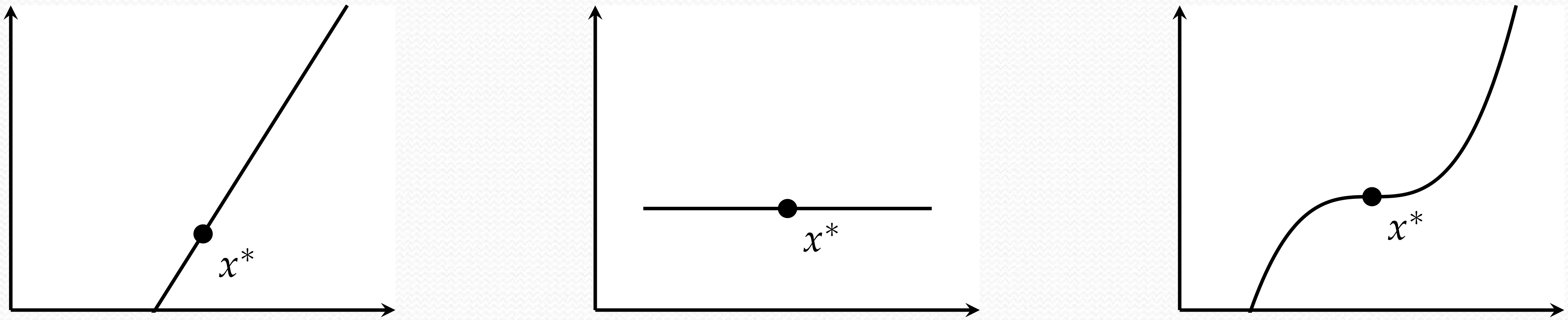
- *Univariate Function*
  - A design point is guaranteed to be at a strong local minimum if the local derivative is zero and the second derivative is positive:
    1. $f'(x*) = 0$
    2. $f''(x*) > 0$
  - A zero derivative ensures that shifting the point by small values does not affect the function value. A positive second derivative ensures that the zero first derivative occurs at the bottom of a *bowl*.

Conditions for Local Minima

- *Univariate Function*

A point can also be at a local minimum if it has a zero derivative and the second derivative is merely nonnegative:

1. $f'(x^*) = 0,$ the first-order necessary condition (FONC)
2. $f''(x^*) \geq 0$, the second-order necessary condition (SONC)

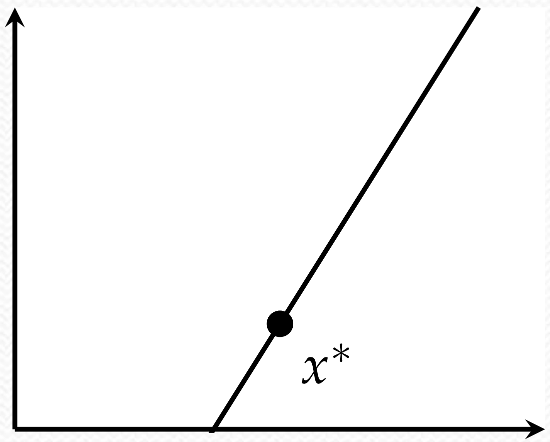


SONC but not FONC — FONC and SONC — FONC and SONC

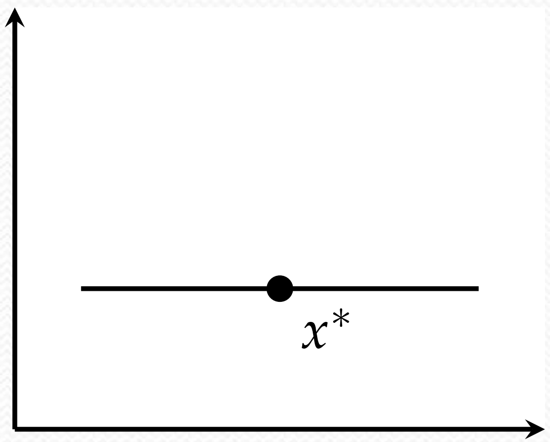
Conditions for Local Minima

- *Univariate Function*

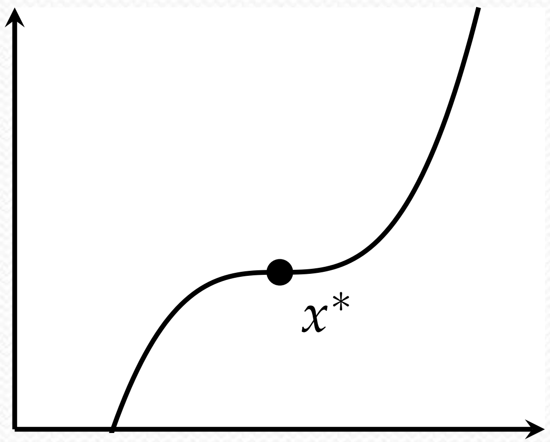
These conditions are referred to as *necessary* because all local minima obey these two rules. Unfortunately, not all points with a zero derivative and a zero second derivative are local minima, as demonstrated in figure.



SONC but not FONC



FONC and SONC



FONC and SONC

Conditions for Local Minima

- *Multivariate Functions*

The following conditions are necessary for x to be at a local minimum of f :

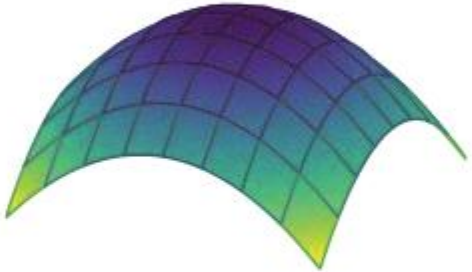
1. , the first-order necessary condition (FONC)
2. $\nabla f(x) = 0$, the second-order necessary condition (SONC)

   $\nabla f(x) \geq 0$

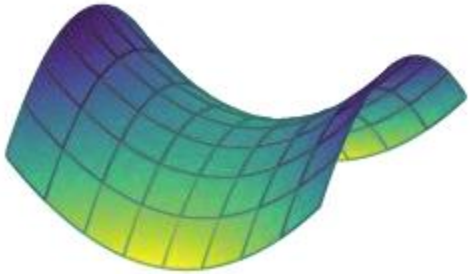
- The FONC and SONC are generalizations of the univariate case. The FONC tells us that the function is not changing at **x**. Next figure shows examples of multivariate functions where the FONC is satisfied. The SONC tells us that **x** is in a bowl.

Conditions for Local Minima

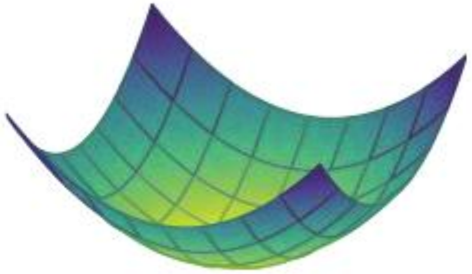
- While necessary for optimality, the FONC and SONC are not sufficient for optimality. For unconstrained optimization of a twice-differentiable function, a point is guaranteed to be at a strong local minimum if the FONC is satisfied and $\nabla^2 f(\mathbf{x})$ is positive definite. These conditions are collectively known as the second-order sufficient condition (SOSC).



A *local maximum*. The gradient at the center is zero, but the Hessian is negative definite.



A *saddle*. The gradient at the center is zero, but it is not a local minimum.



A *bowl*. The gradient at the center is zero and the Hessian is positive definite. It is a local minimum.

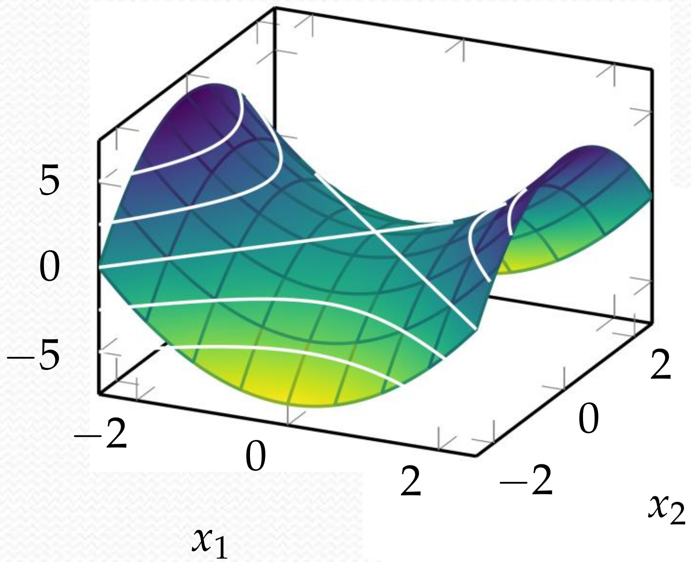
Contour Plots

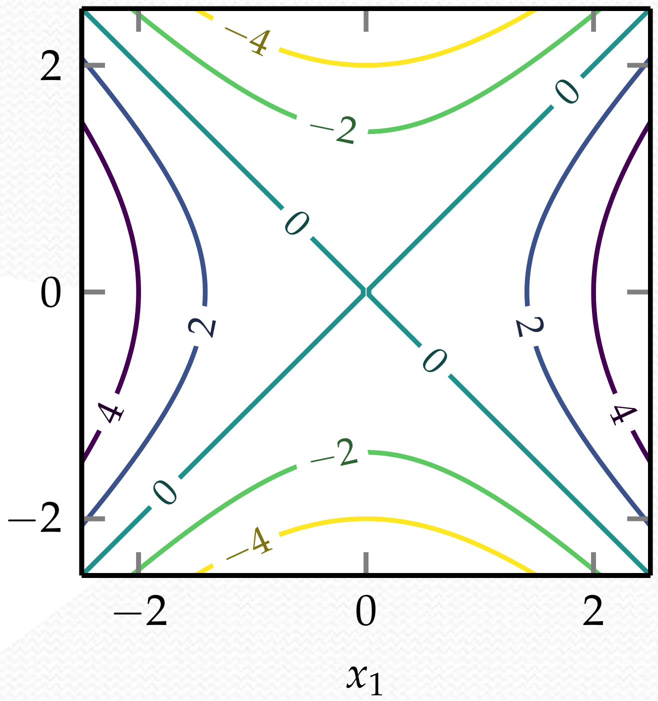
- This lecture will include problems with a variety of numbers of dimensions, and will need to display information over one, two, or three dimensions.
- Functions of the form $f(x_1, x_2) = y$ can be rendered in three-dimensional space, but not all orientations provide a complete view over the domain.
- A contour plot is a visual representation of a three-dimensional surface obtained by plotting regions with constant y values, known as contours, on a two-dimensional plot with axes indexed by x_1 and x_2.

Contour Plots

- $f(x_1, x_2) = {x_1}^2 - {x_2}^2$

This function can be visualized in a three dimensional space based on its two inputs and one output. It can also be visualized using a contour plot, which shows lines of constant y value. A three-dimensional visualization and a contour plot are shown below.





Overview

2. Derivatives → 4. Descent → 5. First-Order, 6. Second-Order

3. Bracketing

7. Direct

8. Stochastic → 9. Population → 20. Expressions

10. Constraints → 11. Linear → 19. Discrete

10. Constraints → 12. Multiobjective

13. Sampling Plans → 14. Surrogate Models → 15. Probabilistic Surrogate Models → 16. Surrogate Optimization

17. Optimization under Uncertainty → 18. Uncertainty Propagation

21. Multidisciplinary Design Optimization

Summary

- Optimization in engineering is the process of finding the best system design subject to a set of constraints
- Optimization is concerned with finding global minima of a function
- Minima occur where the gradient is zero, but zero-gradient does not imply optimality