ME 307 – Machine Elements I

Chapter 3

Deflection Analysis (Part I)

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- **Deflection** is one of the important consideration in the design of machanical elements.
- In some cases, **an element may be strong enough as to carry the loads** without any failure, but **the deflection is so high** that the system may be adversely affected.
- For instance, in the main drive of machine tools, the deflection at the tip of the spindles should not be greater than the tolerable limits. Otherwise, large deflections may create chatter which affects dimensional accuracy and geometrical tolerances *(roundness, cylindricity, concentricity)*.
- As another example, in a power transmission system, the gears are mounted on a solid shaft. If the shaft bends too much, the teeth of the gears can not mesh properly and **the result will be** *noise, wear, and an early failure*.
- In these type of applications, in addition to stress analysis, **deflection at the specific points must be determined** as to finalise the design study.

Deflection under axial loading

When an element is subjected to axial loading, the relationship between deflection and force may be obtained from:

$$\varepsilon = \frac{\delta}{L} = \frac{\sigma}{E} = \frac{F}{AE} \longrightarrow \delta = \frac{FL}{AE} \longrightarrow \frac{F}{\delta} = k = \frac{AE}{L}$$

Resistance of the element against axial deflection

Deflection under torsional loading

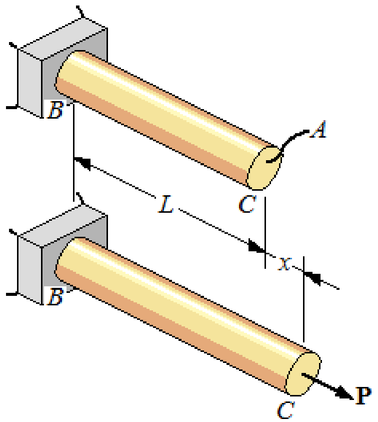
When an element is subjected to torsional loading, the relationship between angular deflection and torque may be obtained from:

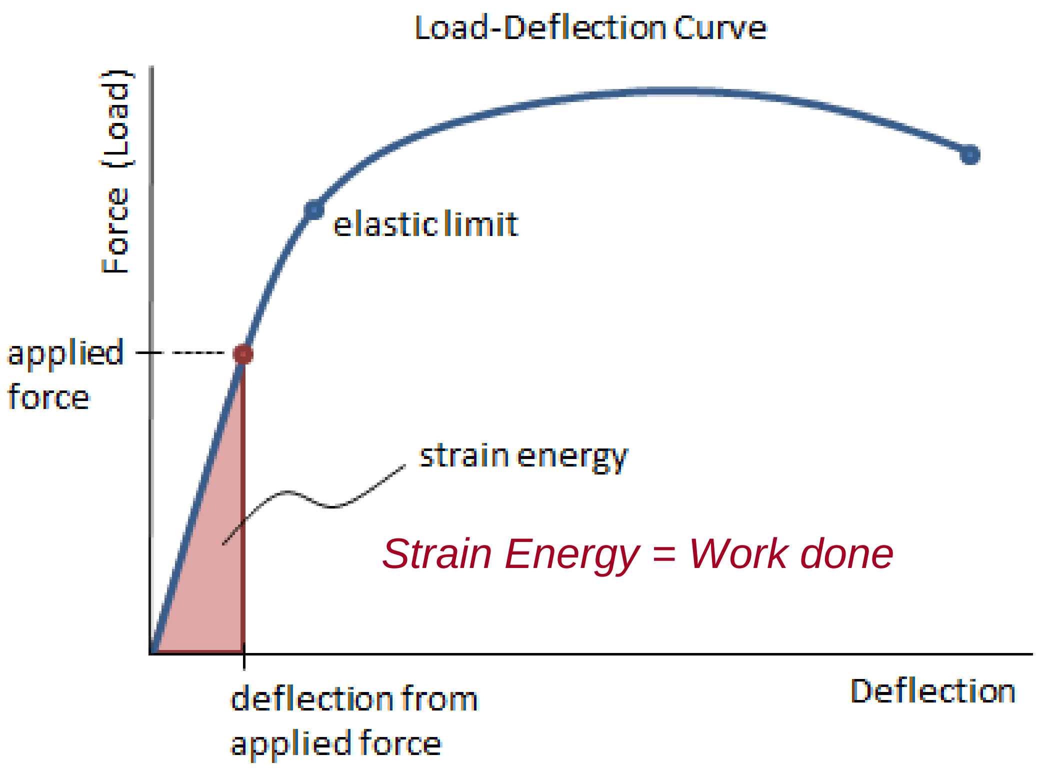
$$\gamma = \frac{\theta r}{L} = \frac{\tau}{G} = \frac{Tr}{GJ} \longrightarrow \theta = \frac{TL}{GJ} \longrightarrow \frac{T}{\theta} = k_t = \frac{GJ}{L}$$

Resistance of the element against torsional deflection

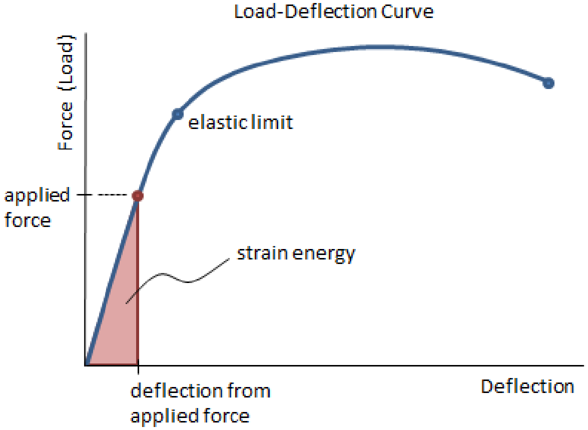
- There are many methods used for the determination of deflection of the elements. **Most known methods are:**
  - *Method of using Singularity Functions*
  - *Double Integration method*
  - *Numerical Integration method*
  - *Graphical Integration method*
  - *Area-Moment method*
  - ***Strain Energy method (Castigliano's Theorem)***
- In this course, **the emphasis is given to the application of strain energy method** because this is a powerful approach to solving a wide range of deflection analysis situations.

- **Strain energy** is the potential energy stored in an element due to elastic deformation caused by different loading conditions.
- Strain energy is dependent on the type of loading *(axial load, torque, bending moment, direct shear or transverse shear)* because the deformation due to the type of loading is different.
- It depends on;
  - The amount of load
  - Type of loading
  - Dimensions





➤ Let's look at the **strain energies** for all types of loading.



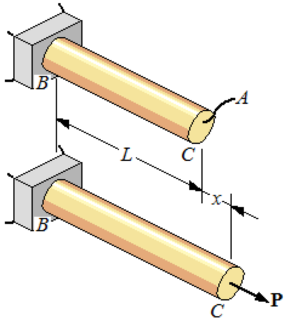
$$U = \frac{Fy}{2} \quad \text{and} \quad y = \frac{F}{k} \longrightarrow U = \frac{F^2}{2k}$$

*This equation is general in the sense that the force F can also mean **axial force**, **torque**, or **moment**.*

For axial loading

$$U = \frac{F^2}{2k} \quad \text{and} \quad k = \frac{AE}{L}$$

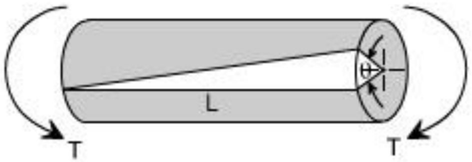
$$\Rightarrow U = \frac{F^2 L}{2AE}$$



For torsional loading

$$U = \frac{F^2}{2k_t} \quad \text{and} \quad k_t = \frac{GJ}{L}$$

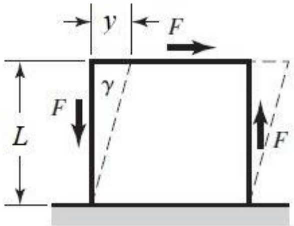
$$\Rightarrow U = \frac{T^2 L}{2GJ}$$



For direct shear

$$U = \frac{Fy}{2} \quad \text{and} \quad y = \frac{FL}{AG}$$

$$\Rightarrow U = \frac{F^2 L}{2AG}$$



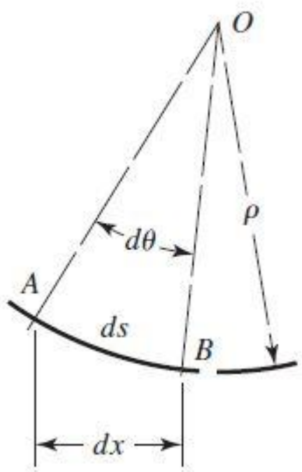
For bending moment

$$dU = \frac{M}{2} d\theta \quad \text{and} \quad \rho d\theta = ds \longrightarrow dU = \frac{Mds}{2\rho}$$

$$\frac{1}{\rho} = \frac{M}{EI} \longrightarrow dU = \frac{M^2 ds}{2EI}$$

For small deflections $\longrightarrow ds \cong dx$

$$\Rightarrow U = \int \frac{M^2 dx}{2EI}$$



For transverse shear

Flexural shear may be subjected to change throughout the length of the element, therefore strain energy is expressed as;

$$\Rightarrow U = \int \frac{CF^2 dx}{2GA}$$

C: 1.2 *for rectangular shape*
C: 1.11 *for circular shape*
C: 2.0 *for tubular shape*

Note: For most of the problems, the contribution of transverse shear on deflection is very small comparing the others and it is usually neglected.

- **As a summary:**

For axial loading $\longrightarrow$ $U = \dfrac{F^2 L}{2AE}$

For torsional loading $\longrightarrow$ $U = \dfrac{T^2 L}{2GJ}$

For direct shear $\longrightarrow$ $U = \dfrac{F^2 L}{2AG}$

For bending moment $\longrightarrow$ $U = \int \dfrac{M^2 dx}{2EI}$

For transverse shear $\longrightarrow$ $U = \int \dfrac{CF^2 dx}{2GA}$ *(usually neglected)*

- **Castigliano's Theorem states that** deflection of a member *at the point of application* and *in the direction of a force* can be found **by taking partial derivative of the total strain energy** with respect to that force.

$$y_i = \frac{\partial U}{\partial F_i}$$

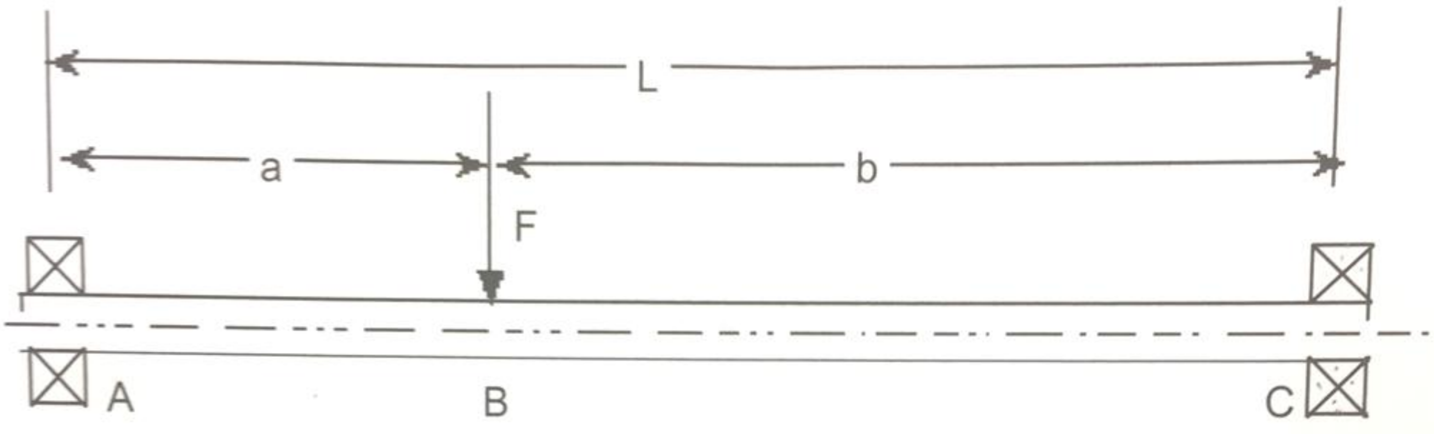
- **The slope of deflection curve** at the point of interest is obtained by taking derivative of total strain energy with respect to bending moment at that point.

$$\theta_i = \frac{\partial U}{\partial M_i}$$

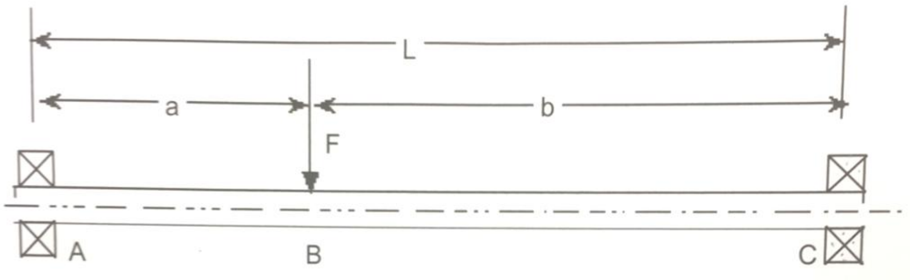
- For Castigliano's theorem to be applied **there must exist a concentrated load at the point under consideration**. If the deflection of a member is required at a point where **no concentrated load is acting**, an imaginary *(fictitious)* force Q is placed at that point and in the resulting deflection expression Q is set to zero.

$$y = \left(\frac{\partial U}{\partial Q} \right)_{Q=0}$$

Consider a simply supported beam as shown below. Using Castigliano's theorem, develop an expression for the deflection at point B.



Solution of Example 3.1



At point B, there is a concentrated load F. Hence, Castigliano's theorem can directly be applied to this case.

$$y_B = \frac{\partial U}{\partial F}$$

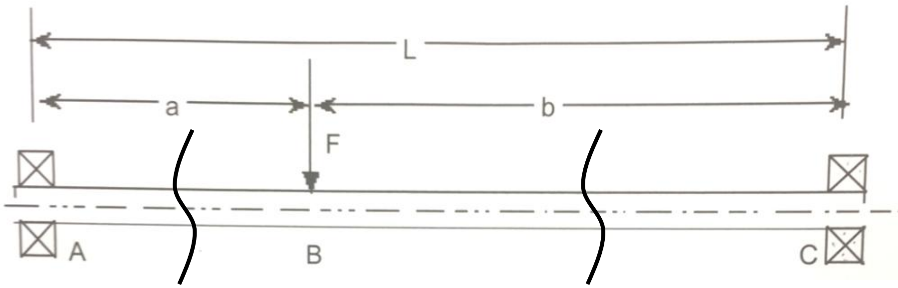
Where U is the total energy which is the sum of strain energies stored in parts AB and BC.

$$U = U_{AB} + U_{BC} \qquad U_{AB} = \int \frac{M_{AB}{}^2 dx}{2EI} \quad \textit{and} \quad U_{BC} = \int \frac{M_{BC}{}^2 dx}{2EI}$$

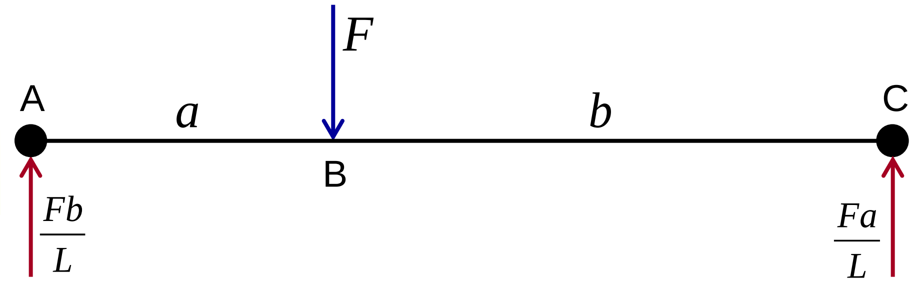
$$y_B = \frac{\partial U}{\partial F} = \frac{1}{EI}\left[\int_0^a M_{AB}\frac{\partial M_{AB}}{\partial F}dx + \int_a^L M_{BC}\frac{\partial M_{BC}}{\partial F}dx\right]$$

M_{AB}, M_{BC} *and* ***partial derivatives of these expressions with respect to F*** *must be determined.*

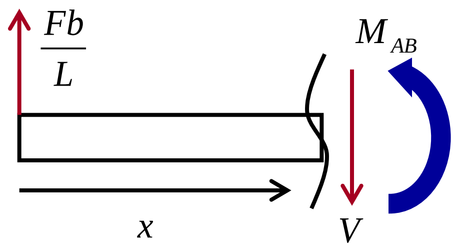
Solution of Example 3.1



Free Body Diagram



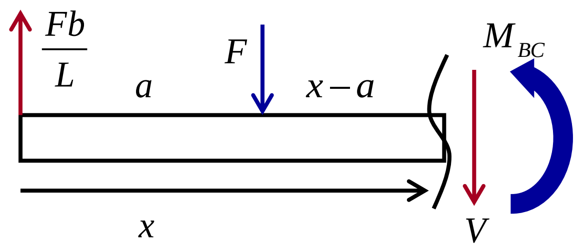
$0 < x \leq a$



$$M_{AB} = \frac{Fb}{L} x$$

$$\frac{\partial M_{AB}}{\partial F} = \frac{b}{L} x$$

$a \leq x < L$

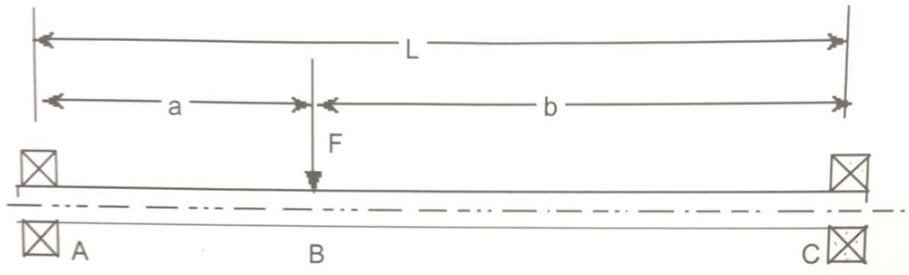


$$M_{BC} + F(x-a) = \frac{Fb}{L} x \longrightarrow M_{BC} = \frac{Fb}{L} x - F(x-a)$$

$$M_{BC} = \frac{Fb}{L} x - Fx + Fa \longrightarrow M_{BC} = F\left(\frac{b}{L} - 1\right) x + Fa$$

$$\frac{\partial M_{BC}}{\partial F} = \left(\frac{b}{L} - 1\right) x + a$$

Solution of Example 3.1



$$M_{AB} = \frac{Fb}{L}x \qquad \frac{\partial M_{AB}}{\partial F} = \frac{b}{L}x \qquad M_{BC} = F\left(\frac{b}{L} - 1\right)x + Fa \qquad \frac{\partial M_{BC}}{\partial F} = \left(\frac{b}{L} - 1\right)x + a$$

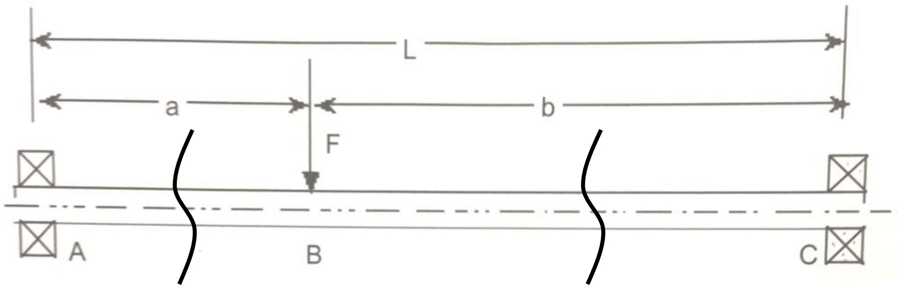
$$y_B = \frac{\partial U}{\partial F} = \frac{1}{EI}\left[\int_0^a M_{AB}\frac{\partial M_{AB}}{\partial F}dx + \int_a^L M_{BC}\frac{\partial M_{BC}}{\partial F}dx\right]$$

$$y_B = \frac{\partial U}{\partial F} = \frac{1}{EI}\left[\int_0^a \frac{Fb}{L}x\frac{b}{L}xdx + \int_a^L \left(F\left(\frac{b}{L} - 1\right)x + Fa\right)\left(\left(\frac{b}{L} - 1\right)x + a\right)dx\right]$$

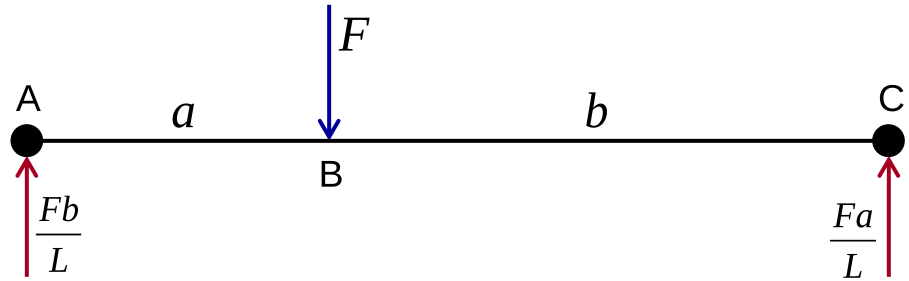
$$y_B = \frac{\partial U}{\partial F} = \frac{Fb^2a^3}{3EIL^2} + \frac{Fa^2b^3}{3EIL^2} = \frac{Fa^2b^2(a+b)}{3EIL^2} = \frac{a^2b^2}{3EIL}F$$

This problem can also be solved by writing moment expression M_{BC} by considering x is changing between 0 and b.

Solution of Example 3.1

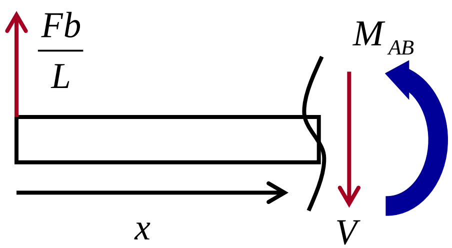


Free Body Diagram



Second Way

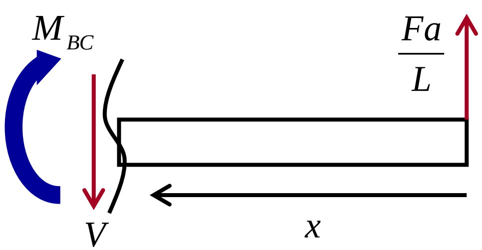
$$0 < x \leq a$$



$$M_{AB} = \frac{Fb}{L}x$$

$$\frac{\partial M_{AB}}{\partial F} = \frac{b}{L}x$$

$$0 \leq x < b$$



$$M_{BC} = \frac{Fa}{L}x$$

$$\frac{\partial M_{BC}}{\partial F} = \frac{a}{L}x$$

$$y_B = \frac{1}{EI}\left[\int_0^a M_{AB}\frac{\partial M_{AB}}{\partial F}dx + \int_0^b M_{BC}\frac{\partial M_{BC}}{\partial F}dx\right]$$

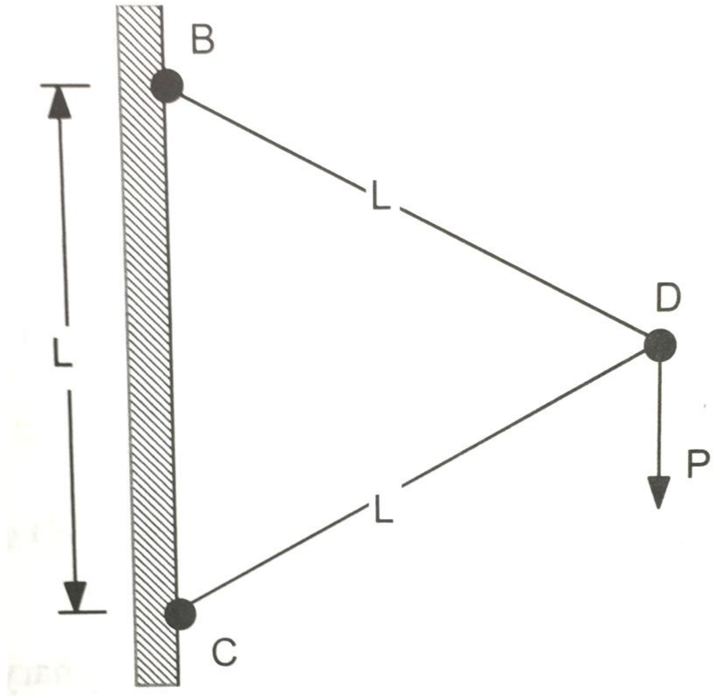
$$y_B = \frac{1}{EI}\left[\int_0^a \frac{Fb^2x^2}{L^2}dx + \int_0^b \frac{Fa^2x^2}{L^2}dx\right]$$

$$y_B = \frac{1}{EI}\left[\frac{Fb^2a^3}{3L^2} + \frac{Fa^2b^3}{3L^2}\right]$$

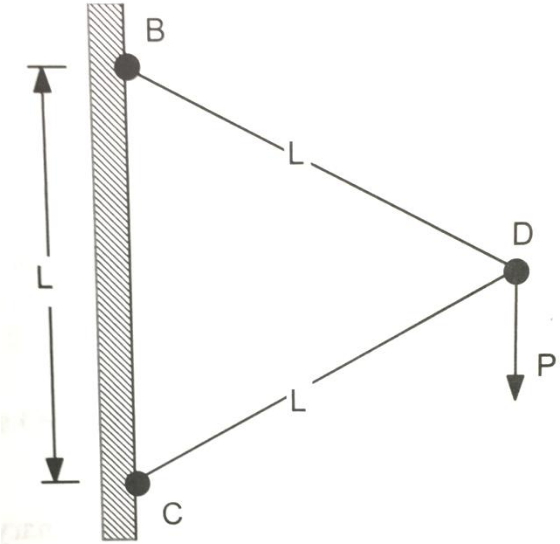
$$y_B = \frac{1}{EI}\left[\frac{Fa^2b^2(a+b)}{3L^2}\right] = \frac{Fa^2b^2}{3EIL}$$

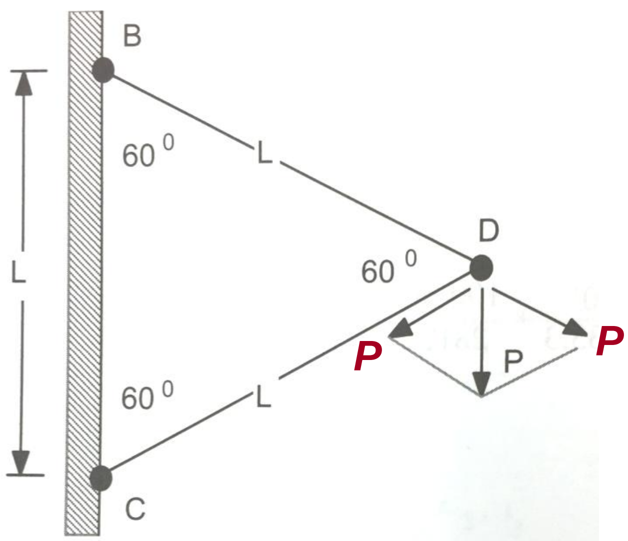
Example 3.3

A simple truss composed of two bars each of length *L* carries a vertical load *P* at joint D. Find the horizontal and vertical components of the total deflection of point D. The bars are made of same material, *DB having a cross sectional area of A* and *DC having a cross sectional area of 2A*. **Use theorem of Castigliano**.



Solution of Example 3.3



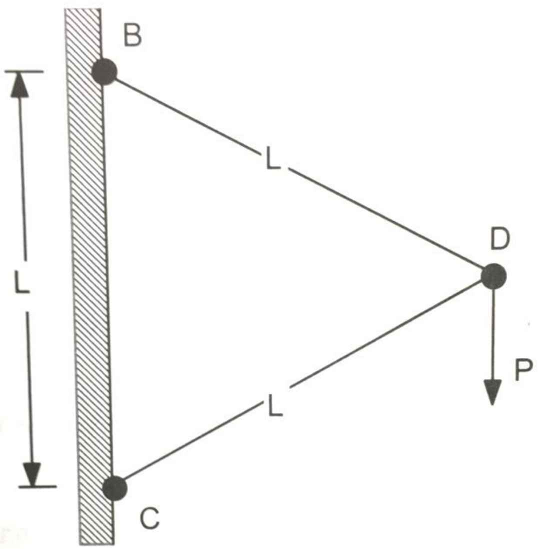


Vertical deflection will be in the direction of the external load P. As shown in Figure, BDC is an equilateral triangle. Force acting on bar DB and DC is equal to P.

$$U_{DB} = \frac{P^2 L}{2AE} \qquad \textit{and} \qquad U_{DC} = \frac{P^2 L}{4AE}$$

$$\delta_v = \frac{\partial\left(U_{DB} + U_{DB}\right)}{\partial P} = \frac{\partial}{\partial P}\left(\frac{P^2 L}{2AE}\right) + \frac{\partial}{\partial P}\left(\frac{P^2 L}{4AE}\right) = \frac{3}{2}\frac{PL}{AE}$$

Solution of Example 3.3

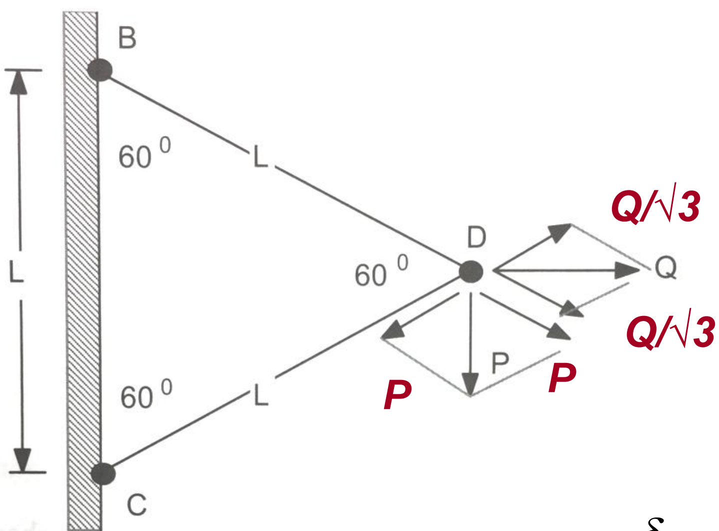


In order to determine the deflection in horizontal direction, we must apply an imaginary (fictitious) force Q at point D as shown in Figure.

Force on DB= $P + \frac{Q}{\sqrt{3}}$ **Force on DC=** $P - \frac{Q}{\sqrt{3}}$

Energies stored in these bars due to the addition of the imaginary force Q can be written as:

$$U_{DB} = \frac{\left(P + \frac{Q}{\sqrt{3}}\right)^2 L}{2AE} \qquad U_{DC} = \frac{\left(P - \frac{Q}{\sqrt{3}}\right)^2 L}{4AE}$$



$$\delta_h = \frac{\partial\left(U_{DB} + U_{DB}\right)}{\partial Q} = \frac{\partial}{\partial Q}\left(\frac{\left(P + \frac{Q}{\sqrt{3}}\right)^2 L}{2AE} + \frac{\left(P - \frac{Q}{\sqrt{3}}\right)^2 L}{4AE}\right)$$

$$\delta_h = \frac{\left(P + \frac{Q}{\sqrt{3}}\right)\frac{1}{\sqrt{3}} L}{AE} + \frac{\left(P - \frac{Q}{\sqrt{3}}\right)\left(-\frac{1}{\sqrt{3}}\right) L}{2AE} = \frac{PL}{2\sqrt{3}AE}$$