Interest

Money → Investment

Money → Depositing in a bank for interest incomes.

Memorize the following WORDS:

WORD

- **Annually** : **once a year (1 times a year)**

  If payments are made more than 1 times a year;
- **Semi-annually** : **2 times a year**
- **Quarterly** : **4 times a year**
- **Monthly** : **12 times a year**
- **Weakly** : **52 times a year**
- **Daily** : **365 times a year**

Interest is the money returned to the owners of capital for use of their capital.

Types of Interest

1) Simple Interest: The amount of capital on which interest is paid is designated as the principal (P).

The rate of interest is defined as the

amount of interest earned by a unit of principal in a unit of time (1 year, 1 month, ...).

e.g., giving someone to use \$1000 for 1 year at a rate of 10% => the interest earned is $1000 \times 0.10 = \$100$ after 1 year.

* Simple interest requires compensation payment at a constant interest rate based on the original principal.

e.g., \$1000 loaned for 4 years at a constant interest rate of 10% per year =>

$1000 \times 0.10 \times 4 = \$400$ interest income.

$\therefore$ $\boxed{Z = P \times i \times n}$

Z: amount of simple interest during n interest period

P: principal (Capital)

i: interest rate based on the length of one interest period.

n: # of time units or interest period.

$S = P + Z = P + P \times i \times n \Rightarrow$

$\boxed{S = P(1 + i \times n)}$ ⇝ Simple interest

S = principal + simple interest paid during n periods.

2) Ordinary and Exact Simple Interest:

- In general, time unit is 1 year (i.e., n is based on 1 year) $\Rightarrow S = P \times (1 + i \times n)$
- But, if interest period is less than 1 year,

$\Rightarrow$ ⊛ Ordinary simple interest $\Rightarrow$

$$30 \frac{\text{day}}{\text{month}} \times 12 \frac{\text{month}}{\text{year}} = 360 \text{ days/year}$$

$$\text{Ordinary simple interest} = P \times i \times \underbrace{\frac{d}{360}}_{\text{new } n}$$

$\Rightarrow$ ⊛ Exact simple interest $\Rightarrow$ 365 days in a normal year.

$$\text{Exact simple interest} = P \times i \times \underbrace{\frac{d}{365}}_{\text{new } n}$$

d: # of days in an interest period.

Example: What is the ordinary interest on $1360 for 90 days at 4% annual interest rate ?

Solution:

Given: P = $1360, i = 0.04, d = 90 days,

Z = P*i* (d/360) = $1360 x0.04* 90/360 = $13.60

Example: Find the exact interest on $500 at 8% annual interest rate for 45 days.

Solution:

Given: P = $500, i = 0.08, d = 45 days

Z = P*i* (d/365) = $500 x 0.08 x 45/365 = $4.93

3) Compound Interest

Simple interest has the property that the interest earned is not invested to earn additional interest.

In contrast, compound interest has the property that the interest earned at the end of one period is automatically invested in the next period to earn additional interest.

e.g., Initial loan of $1000 at 10% interest rate annually $\Rightarrow$

- after 1 year, the payment $= 1000(1+0.1\times 1) = 1100$

- if this payment is not made $\Rightarrow$

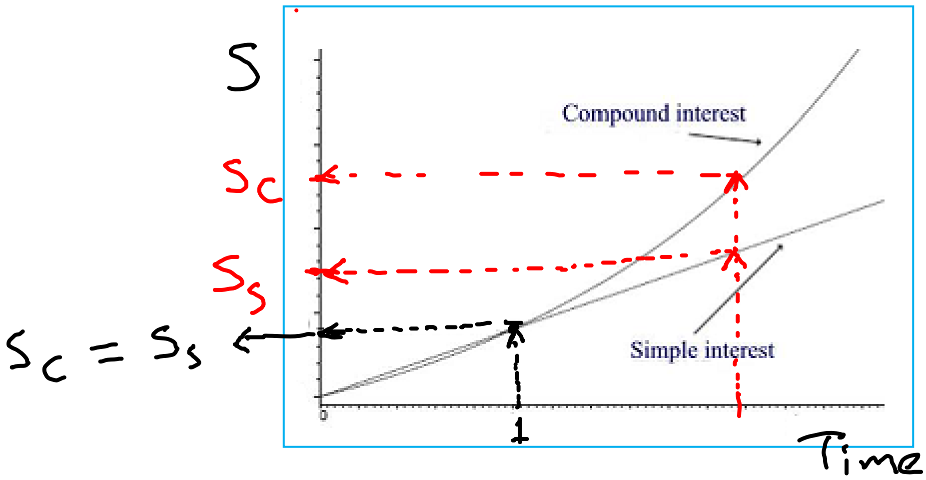
the interest for 2^{nd} year $= (1100) \times 0.1 = \$110$

and the total compound amount due after 2 years is

$$\underbrace{\$1000}_{\text{principal}} + \underbrace{\$100}_{\text{after } 1^{st} \text{ year}} + \underbrace{\$110}_{\text{after } 2^{nd} \text{ year.}} = \$1210$$

$$\boxed{S = P(1+i)^n} \rightsquigarrow \text{compound interest.}$$

$$S = 1000(1+0.10)^2 = \$1210.$$



$S_c > S_s$ at a fixed time.

Simple and compound interest produce the same result over one measurement period. Compound interest produces a larger return than simple interest for periods greater than 1 and smaller return for periods smaller than 1 as shown in the Figure.

Nominal and Effective Interest Rates:

- Interest rate is only meaningful in the context of time, in general is understood as per year, which may be called the nominal interest rate.

  It is the rate unadjusted for inflation.
- With other periods of time than year, like month, week or day, the interest rate may be called the effective interest rate.

Real Interest Rate = Nominal Interest Rate – Inflation Rate

e.g.) 100 TL investment with 10% nominal interest rate $\Rightarrow$

$100(1+0{,}1\times1) = 110$ TL return.

If there is 7% inflation in the country $\Rightarrow$

Real int. rate $= 10 - 7 = 3\%$ $\Rightarrow$

$100\times(1+0{,}03\times1) = 103$ TL total return.

$110 - 103 = 7$ TL value loss.

Effective Annual Interest Rate:

If i is the nominal interest rate under conditions where there are m conversions (# of compounding periods) or interest periods per year $\Rightarrow$

the interest rate based on the length of one interest period is i/m and the amount S after 1 year is

$$\boxed{S = P\left(1 + \frac{i}{m}\right)^m}$$

For more than 1 year, i.e., n years $\Rightarrow$

$$\boxed{S = P\left(1 + \frac{i}{m}\right)^{n \times m}}$$

⊗ If i_{eff} = effective interest rate, then,

$$\boxed{i_{eff} = \left(1 + \frac{i}{m}\right)^m - 1}$$

The effective annual interest rate is higher than the nominal rate.

e.g., depositing \$1000 in a bank, interest rate 10% annually $\Rightarrow$ after 1 year you will receive $1000 \times 0.1 = \$100$ interest and the total return $1000 + 100 = 1100$.

In this example, the effective interest rate is the same as the quoted interest 10%.

Now, if the bank pays the money semi-annually (the compounding will occur twice a year) $\Rightarrow$

This means that when the first payment occurs the bank will pay you $\underbrace{0.05}_{5\%} \times 1000 = \$50$ interest.

$\Rightarrow 1000 + 50 = 1050$ return.

At the end of second payment (2^{nd} six month).

$0.05 \times 1050 = \$52.50$.

The total return $= 1000 + 50 + 52.50 = 1102.50$

OR Total return $= S = 1000 \times \left(1 + \frac{0.10}{2}\right)^{1 \times 2} = 1102.50$

Semiannually $\Rightarrow$ WORD $= 2$

Here, the effective interest rate would be 10.25%.

It is more than the quoted interest rate 10%.

$\therefore$ The effective annual interest rate is the annual interest rate that accounts for the effect of compounding.

For example, let's assume you have a loan with 8.25% nominal interest rate. When you make monthly payments (12 compounding periods), your effective annual interest rate will be

$$\left(1 + \frac{0.0825}{12}\right)^{12} - 1 = 8.57\%.$$

* The nominal rate is calculated as you multiply the # of periods by the interest rate for the period.

e.g., If monthly interest rate is 1%, what is the nominal interest rate?

nominal interest rate = 12 × 1% = 12%

Example: It is desired to borrow $1000 from an agency at a monthly interest rate of 2 %. Determine the following:

a) The total amount of principle plus simple interest due after 2 years if no intermediate payments are made.
b) The total amount of principle plus compound interest due after 2 years if no intermediate payments are made.
c) The nominal interest rate when the interest is compunded annually.
d) The effective annual interest rate when the interest compunded monthly.

Solution:

a) $P = \$1000$, $i = 2\%$ monthly, $n = 24$ interest period in 2 years.

$$S = P(1 + i \times n) = 1000\,(1 + 0{,}02 \times 24) = \$1480$$

b) $S = P(1+i)^n = 1000\,(1+0{,}02)^{24} = \$1608$

c) Nominal interest rate $= i = 12 \times 2\% = 24\%$ or

$i = 0{,}24$

d) # of interest period $m = 12$, $i = 0{,}24$

$$i_{eff} = \left(1 + \frac{i}{m}\right)^m - 1 = \left(1 + \frac{0{,}24}{12}\right)^{12} - 1 = 0{,}268 \text{ or}$$

26.8%.

4) Continuous Interest:

Continuous interest is a form of compound interest. With continuous interest, the length of the compounding period is small (month, day, hr, etc.). Continuously compounded interest means that your principal is continuously and constantly earning interest and the interest keeps earning on the interest earned.

$\boxed{S = P \times e^{i \times n}}$ ⇝ continuous interest

$$i_{eff} = e^{i} - 1$$

$$S = P(1 + i_{eff})^{n}$$

i: nominal interest rate

n: periods (# of years).

<u>Example</u>: If $4000 is invested at an annual rate of 6 % compounded continuously, what will be the final value of the investment after 10 years ?

<u>Solution</u>:

$$S = 4000 \times e^{(0.06 \times 10)} = \$7288.48$$

Example: For the case of a nominal annual interest rate of 20 % determine,

a) The total amount to which $1.0 of initial principal would accumulate after one 365-day year with daily compounding.
b) The total amount to which $1.0 of initial principal would accumulate after one year with continuous compounding.
c) The effective annual interest rate if compounding is continuous.

Solution: a) $P = \$1.0$, $i = 0.20$, $m = 365$

$$S = P\left(1 + \frac{i}{m}\right)^{m} = 1 \times \left(1 + \frac{0{,}20}{365}\right)^{365} = \$1.2213$$

b) $n = 1$ year

$$S = P \times e^{i \times n} = 1 \times e^{(0{,}20 \times 1)} = \$1{,}2214$$

c) $i_{eff} = e^{i} - 1 = e^{0{,}20} - 1 = 0{,}2214$ or 22.14%.

FUTURE VALUE and PRESENT VALUE of MONEY

In this section, we discuss the question of determining the today's or present value of some amount in the past or in the future. Unless stated otherwise, we assume in this section that we are in a compound interest situation.

How do we know what to choose ?

- $100 today or $108 in the future

- **$97 today or $102 in the future**
- **$100 today or $120 in the future**
- **$1000 today or $1050 in the next year**

What is Future Value ?

Future value is the value that a some of money today will be worth at some point in the future if invested for a return.

S FV = $PV(1+i)^n$ P

FV: Future value (money in the future)

PV: Present value (amount of money today)

i: interest paid by the investment (also called the discount rate or the required rate of return)

n: number of periods the investment will be held.

What is Present Value ?

Present value is today's value of a future cash flow.

- **A dollar paid to you one year from now is less valuable than a dollar paid to you today.**

i.e., $100 today > $100 received in the future (because of inflation etc.)

$$PV = \frac{FV}{(1+i)^n}$$

- **i: interest rate (% divided by WORD)**

e.g., 6 % annual interest compounded semi-annually;

The WORD is semi-annually = 2

i = 0.06/2 =0.03

e.g., 6 % annual inerest compounded monthly (WORD=12)

i = 0.06/12

e.g., 6 % annual interest compounded weekly (WORD = 52)

i = 0.06/52

e.g., 6 % annual interest compounded daily (WORD = 365)

i = 0.06/365

- **n: # of compounding period (years x WORD)**

e.g., compounded semi-annually for 10 years,

WORD (semi-annually) = 2, year = 10

n = 10x2 = 20

e.g., compounded monthly for 5 years,

WORD (monthly) =12, year = 5

n = 5x12

Example: Suppose you have opportunity to make an investment that will earn 10 % annually. What is the best financial decision: $1000 today or $1050 in the next year ?

Solution:

$$\left.\begin{array}{l} FV = \$1050 \\ i = 0.10, (10\%) \\ n = 1 \end{array}\right\} \quad FV = PV \cdot (1+i)^n \quad \Rightarrow$$

Calculate future value of $1000 if

you make investment at 10% annual interest rate.

$FV = 1000(1+0.1)^1 = \$1100 \Rightarrow$

We should choose \$1000 now instead of \$1050 in the next year.

Example: **Assume that you would like to put money in an account today to make sure your child has enough money in 10 years to buy a car. If you would like to give your child \$10,000 in 10 years, and you know you can get 5% interest per year from a savings account during that time, how much should you put in the account now?**

Solution:

Present value to be calculated $\Rightarrow$

$$PV = \frac{10000}{(1+0.05)^{10}} = \$6139.13$$

Thus, \$6,139.13 will be worth \$10,000 in 10 years if you can earn 5% each year. In other words, the present value of \$10,000 in this scenario is \$6,139.13.

Example: **\$1000 invested at 3 % interest per year, compounded monthly for 5 years. What is the future value of money, i.e., how much money will be in the account ?**

Solution:

$FV = PV(1+i)^n, \quad i = \frac{0.03}{12}, \quad n = 5 \times 12$

$= 1000\left(1 + \frac{0.03}{12}\right)^{5\times 12} = \$\boxed{1161.6}$ after 5 yrs.

How much interest did you make?

Amount of interest made = $\boxed{1161.6 - 1000}$

$= \$\boxed{161.6}$

Example: Invest \$2000 at 12 % annual interest rate compounded semi-annually for 2 years. What is FV ?

Solution:

$i = \frac{0.12}{2}, \quad n = 2 \times 2 \Rightarrow$

$FV = 2000 \times \left(1 + \frac{0.12}{2}\right)^{2\times 2} = \$2524.95$

Example: Invest \$10000 at 12 % annual interest rate compounded quarterly for 1 year. What is FV ?

Solution:

$i = \frac{0.12}{4 \rightsquigarrow \text{quarterly}}, \quad n = 1 \times 4 \Rightarrow$

$FV = 10000 \times \left(1 + \frac{0.12}{4}\right)^{1\times 4} = \$11255.09$

Example: Invest \$100 at 10 % interest per year compounded annually for 2 years. What is FV ?

Solution:

$i = \frac{0.10}{1}, \quad n = 2 \times 1 = 2 \quad$ (WORD = 1, year = 2)

$$FV = 100\left(1 + \frac{0.10}{1}\right)^{2 \times 1} = \$121.$$

ANNUITIES

Annuities are equal, periodic outflows/inflows at regular intervals, e.g. rent, mortgage, car loan, and retirement annuity payments.

- **An annuity stream can begin at the start (beginning) of each period (annuity due) as is true of rent and insurance payments or at the end of each period (ordinary annuity), as in the case of mortgage and loan payments.**

<u>Annuity: Payments Made at The Beginning of Each Period</u>

To calculate the future value of an annuity due use the following formula:

$$FV = C \times \left[\frac{(1+i)^n - 1}{i}\right] \times (1+i)$$

FV: future value

C: equal payment or deposit amount

i: interest rate (as decimal)

n: # of payments

<u>Example</u>: Suppose you want to save money for your child's college expenses. Let's suppose you deposit $1000 at the beginning of each year, for 18 years, at an annual interest

rate of 5 %. How much is available for your child when she/he starts school ?

<u>Solution:</u>

$C = \$1000, \quad i = 0.05, \quad n = 18 \implies$

$$FV = 1000\left[\frac{(1+0.05)^{18}-1}{0.05}\right] \times (1+0.05) = \underbrace{\$29539}_{\text{after 18 yrs.}}$$

<u>Example</u>: Suppose for an annuity due, you want to have \$30000 in the bank after 20 years. Assuming you make deposits at the beginning of each year at an annual interest rate of 4 %. How much would you have to deposit at the start of each year assuming each deposit is the same ?

<u>Solution</u>:

$$30000 = C \times \left[\frac{(1+0.04)^{20}-1}{0.04}\right] \times (1+0.04) \implies$$

$C = \$968.70$ to be deposited at the beginning of each year.

<u>Annuity: Payments Made at The End of Each Period (ordinary annuity)</u>

The formula for calculating the future value of an ordinary annuity stream is as follows:

$$FV = C \cdot \left[\frac{(1+i)^n - 1}{i}\right]$$

C: equal payment or deposit at the end of each period.

Example: How much money will you accumulate by the end of year 10 if you deposit $3,000 each for the next ten years in a savings account that earns 5% interest per year?

Solution:

$$FV = 3000 \times \left[\frac{(1+0.05)^{10} - 1}{0.05} \right] = \$37733.68$$

Example: Suppose you would like to have $25,000 saved 6 years from now to pay towards your down payment on a new house. If you are going to make equal annual end-of-year payments to an investment account that pays 7%, how big do these annual payments need to be?

Solution:

$$25000 = C \times \left[\frac{(1+0.07)^{6} - 1}{0.07} \right] = C \times 7.1533$$

$C = \$3494.89$ to be paid annually.

Example: In 20 years, you are hoping to have saved $100,000 towards your child's college education. If you are able to save $2,500 at the end of each year for the next 20 years, what rate of return must you earn on your investments in order to achieve your goal?

Solution:

$$100000 = 2500 \times \left[\frac{(1+i)^{20} - 1}{i} \right] \Rightarrow i = 0.0677 \text{ or } 6.77\%.$$

The Present and Future Values of an Ordinary Annuity

The present value of an ordinary annuity measures the value today of a stream of cash flows occurring in the future.

$$FV = C \times \left[\frac{(1+i)^n - 1}{i} \right]$$

Remember that $PV = \frac{FV}{(1+i)^n}$

$$PV = \frac{C \times \left[\frac{(1+i)^n - 1}{i} \right]}{(1+i)^n} = C \times \left[\frac{(1+i)^n - 1}{(1+i)^n \times i} \right]$$

divide numerator and denominator by $(1+i)^n \Rightarrow$

$$PV = C \times \left[\frac{1 - \frac{1}{(1+i)^n}}{i} \right]$$

<u>Example</u>: What is the value today and future value equivalent of receiving $3,000 every year for the next 30 years if the interest rate is 5% annually?

<u>Solution</u>:

If I know its future value, I can compute its present value.

$$FV = 3000 \times \left[\frac{(1+0.05)^{30} - 1}{0.05} \right] = \$199316.54$$

$$PV = \frac{FV}{(1+i)^n} = \frac{199316.54}{(1+0.05)^{30}} = \$46117 \quad \text{OR}$$

PV can be found from $PV = C \times \left[\frac{1 - \frac{1}{(1+i)^n}}{i} \right]$

Example: What is the present value of an annuity of $10,000 to be received at the end of each year for 10 years given a 10 % discount rate?

Solution:

$$PV = 10000 \times \left[\frac{1 - \frac{1}{(1+0.1)^{10}}}{0.10} \right] = \$ 61445$$

Example: Suppose you plan to get a $9,000 loan from a furniture dealer at 18% annual interest with annual payments that you will pay off in over five years. What will your annual payments be on this loan?

Solution:

$$PV = C \times \left[\frac{1 - \frac{1}{(1+i)^n}}{i} \right] \Rightarrow$$

$$9000 = C \times \left[\frac{1 - \frac{1}{(1+0.18)^5}}{0.18} \right] \Rightarrow$$

$$C = \$ 2878$$

Example: You have just found the perfect home. However, in order to buy it, you will need to take out a $300000, 30 year mortgage at an annual rate of 6 percent. What will your monthly mortgage payments be?

Solution:

WORD: Monthly = 12, year = 30 =>

$i = \frac{6\%}{12} = 0.5\%$, $n = 30 \times 12 = 360$

$$300000 = C \times \left[\frac{1 - \frac{1}{(1+0.005)^{360}}}{0.005} \right] \Rightarrow$$

C = $1798.66 to be paid monthly.

Your project is feasible or NOT?

Compare your Total Capital Inv. with

10% interest rate for 10 years.

Assume Compound Interest.

After 10 years;

If your net profit > return due to interest

⇒ The project may be FEASIBLE.

Anüite (Annuity), belirli bir zaman süreci içerisinde, eşit aralıklarla verilen veya alınan eşit ödemeler serisidir.

Anüiteler, ödemeler serisinin başlama noktasına göre, dönem başı veya dönem sonu olarak ikiye ayrılır.

Dönem Sonu Anüitelerin Gelecek Değeri

Her devre sonu alınacak veya verilecek eşit taksitlerin, belirli bir süre sonunda ulaşacağı değer, şöyle hesaplanır:

$FVA_n = P\ [((1 + i)^n - 1) / i\ ]$

FVA_n**= Anüitenin n dönem sonundaki gelecek değeri**

P = Eşit aralıklarla yatırılan (veya çekilen) eşit para turarı

i=Faiz oranı, n=Dönem sayısı

Örnek: Bir yatırımcı, %50 faiz üzerinden, her yıl sonunda 4 yıl boyunca, 1.000.000 TL yatırırsa, 4. yılın sonundaki yatırım tutarı ne kadar olur?

$FVA_n = P\ [(1 + i)^n - 1) / i\ ]$

$FVA_n = 1.000.000\ [(1+0.50)^4 - 1 / 0.50] = 8.125.000$ **TL olur.**

<u>Dönem Sonu Anüitelerin Şimdiki Değeri</u>

Her yıl sonunda yatırılan veya alınan eşit tutarların bugünkü değeridir.

$PVA = P.\ [\ 1 - (1/(1+i)^n)]/i$

Örnek: 4 yıl boyunca, her yıl sonunda elde edilen 100.000 TL'nin, %30 faiz oranı üzerinden bugünkü değeri kaç TL'dir?

$PVA = P.\ [[\ 1 - 1/(1+i)^n]/i]$

$PVA = 100.000\ [[1 - 1/(1+0,30)^4]/0,30] = 216.620$ **TL**

<u>Dönem Başı Anüitelerin Gelecek Değeri</u>

Eşit aralıklarla yapılan eşit ödemeler, her dönem başında yapılıyorsa, buna peşin anüite denir.

Peşin anüite şöyle hesaplanabilir:

$FVA_n = P\ [((\ 1 + i\ )^n - 1) / i\ ]\ (\ 1 + i\ )$

FVA_n**= Anüitenin n dönem başındaki gelecek değeri**

P = Eşit aralıklarla yatırılan eşit para turarı

i=Faiz oranı,   n=Dönem sayısı

Örnek: Bir yatırımcı, %50 faiz üzerinden, her yıl başında 4 yıl boyunca, 1.000.000 TL yatırırsa, 4. yılın sonundaki yatırım tutarı ne kadar olur?

$$FVA_n = P\ [((\ 1 + i\ )^n - 1)\ /\ i\ ]\ (\ 1 + i\ )$$

$$FVA_n = 1.000.000[((1+0.50)^4-1)/0.50](1+0.50) = 12.187.500 \text{ TL olur.}$$

<u>Dönem Başı Anüitelerin Şimdiki Değeri</u>

Her dönem başında, eşit aralıklarla ödenen veya alınan eşit taksitlerin şimdiki değerinin hesaplanmasıdır.

$$PVA = P.\ [\ (1+i)\ [(1+i)^n - 1\ /(1+i)^n\ -1]]$$